Module 1

Mathematical tools

Introduction

This module attempts to summarise basic concepts in mathematics that underlie decision-making tools. The backgrounds and needs of students may differ vastly, which has made this a difficult module to construct.

The idea is not to make students experts in abstract mathematics, but to enable them to participate efficiently in a management environment that is becoming more knowledge-driven and quantified each year.

Upon completion of this module the student will be able to:

Outcomes

- **use** and **apply** basic mathematical operations;
- **understand** sequences and series of numbers (with application to compound interest rates, present value, future value and internal rate of return);
- **solve** linear equations and simultaneous linear equations (with application to supply, demand, break-even and budget constraints);
- **understand** and **apply** non-linear functions; and
- **understand** the roles of differentiation and integration in modelling (with application to optimisation, elasticity, surplus, growth rate and marginal cost).

Unit 1

Basic operations with numbers

Outcomes

Upon completion of this unit the student will be able to:

- **explain** the different number systems;
- **apply** various arithmetic operations to numbers (such as percentages and ratios);
- **explain** the rounding-off procedure;
- **identify** arithmetic and geometric sequences; and
- **apply** sequences and series of numbers to business situations.

Number systems and operations

Rounding off numbers

It is important to get students to realise the importance of the rounding-off phenomenon.

- Calculators and computer software round off numbers to a certain number of decimals. These differ from calculator to calculator, with the result that answers to questions can differ slightly.
- To ensure accuracy when doing calculations, as many decimal places as is reasonable should be kept up to the final answer. At that stage, a decision has to be made as to the required accuracy of the answer and this will depend on the context.
- Monetary amounts have two decimal places and quantities such as number of items should be whole numbers. It is not sensible to talk about 12.842 books or GBP 12,000.7281436. The rounded figures should be 13 books and GBP 12,000.73

Case study: EduBooks

Consider the following financial statements for EduBooks, which employs 40 people. Figures are given in pounds (millions).

Then do Activity 1.3 at the end of the statements.

Balance Sheet for EduBooks			
December 2009			
Assets		**Liabilities and Shareholders' Equity**	
Current assets		Current liabilities	
Cash	10.355	Accounts payable	18.900
Inventories	22.561	Income taxes	1.250
Total current assets	32.916	Total current liabilities	20.150
Property and equipment	55.765	Long-term debt	15.200
Less depreciation	12. 879		
Property and equipment, net	42.886	Total shareholders' equity	40.452
Total assets	75.802	Total liabilities and shareholders' equity	75.802

The balance sheet formula is: Assets = Liabilities + Shareholders' Equity

Income Statement for EduBooks		
For years ending 2005 and 2006		
	2005	2006
Net sales	*	*
Cost of sales	(2.900)	(3.485)
Gross income	15.255	18.990
Operating expenses	(2.800)	*
Income Statement for EduBooks		
For years ending 2005 and 2006		
Operating income	12.455	
Interest expense	(1.006)	(1.006)
Net profit before tax	*	*
Tax	*	*
Net income	*	*

Activity 1.3

Activity
Calculate for EduBooks

What will you do?

Calculate the following for the company:

1. Complete the income statement by filling in all cells marked *. (Assume a constant tax rate over 2005–2006 of 22% on net profit. For 2006, assume a 10% increase in operating expenses.)
2. Investigate the company's liquidity by calculating the current ratio from the balance sheet.
3. What percentage of total current assets is cash?
4. What percentage of total liabilities does shareholder's equity form?
5. What is the ratio of long-term debt to equity?
6. Measure the personnel productivity of the company by calculating the revenue per employee.
7. What is the increase or decrease in net income from 2005 to 2006?
8. If the discount rate is 7.3% p.a. for 2009, what would the present value of the total assets have been on 1 January 2009

Solutions

1.

Income Statement for EduBooks		
For years ending 2005 and 2006		
	2005	2006
Net sales	18.155	22.475
Cost of sales	(2.900)	(3.485)
Gross income	15.255	18.990
Operating expense	(2.800)	(3.08)
Operating income	12.455	15.91
Interest expense	(1.006)	(1.006)
Net profit before tax	11.449	14.904
Tax	2.519	3.279
Net income	8.93	11.625

2. Current ratio is an indication of a company's ability to meet short-term debt obligations. It is the ratio of current assets to current liabilities. The higher the ratio, the more liquid the company is. If the current assets of a company are more than twice the current liabilities, then that company is considered to have good short-term

financial strength. In this case, Current ratio = 32.916/20.150 = 1.634. The company is doing reasonably well.

3. Percentage cash = 10.355/32.916 = 0.3146 (rounded off) = 31.46%
4. Total liabilities: 75.802; Shareholder equity: 40.452

   Percentage = 40.452 / 75.802 = 0.5337 (rounded off) = 53.37%
5. Ratio: 15.2 : 40.452 = 15.2 / 40.452 = 0.3758
6. Year 2005 Revenue per employee = Net sales / number of employees

   = 18.155 / 40 = 0.453875 million

   = 453,875 pounds

   Year 2006 Revenue per employee = Net sales / number of employees

   = 22.475 / 40 = 0.0.561875 million

   = 561,875 pounds

   These are excellent figures and show increasing productivity.
7. Change in net income: 11.625 – 8.93 = 2.695 million pounds
8. Present value of total assets on 1 January 2009:

   75.802 / (1 + 0.073) = 70.645 million pounds

Sequences and series of numbers

Trends and limits for sequences

Trends and limits play an important role in analysis and forecasting. The lecturer may want to expand on this. Some students may have done calculus and will know the meaning of lim S_n as $n \to \infty$.

Series

Sums of series play a role in determining present values, internal rates of return, annuities, and so on.

Note it!

Formulas for sums of finite arithmetic and geometric series

Arithmetic sequence: $S_n = \frac{n}{2}[2T_1 + (n-1)d]$

Geometric series: $S_n = \frac{T_1(1-r^n)}{1-r} =. \frac{T_1(r^n - 1)}{r-1}$

Activity 1.6

Activity

Identify sequences and discuss trends

What will you do?

1. Write down the first six terms of each sequence from the formula for these general terms.

   Discuss trends, if any, in the numbers.

   a) $T_n = 2n - 5$

   b) $T_n = (2n)^{-1}$

   c) $T_n = 3T_{n-1} + 2$, with $T_1 = 1$

2. Identify the sequences, where possible, and give the expression for the general term in each of these cases.

   Discuss any trends in the numbers.

   a) 5, 8, 11, 14…

   b) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \ldots$

   c) 3, 9, 27, 81…

3. Sales data has been collected over 12 months and the pattern is forecast to continue into the future. (The numbers are in thousands.)

   Sales data:

   1.5, 3.5, 5.5, 7.5, 9.5, 11.5, 13.5, 15.5, 17.5, 19.5, 21.5, 23.5

   a) Determine a formula to describe the sales in a general month.

   b) Use it to predict the sales in month 48.

   c) Find the total of all sales after 60 months.

Solutions

1. a) −3, −1, 1, 3, 5, 7

   b) $\frac{1}{2}; \frac{1}{4}; \frac{1}{6}; \frac{1}{8}; \frac{1}{10}; \frac{1}{12}$

   c) 1, 5, 17, 53, 161, 485

2. a) Arithmetic sequence with unbounded increasing tendency (limit = ∞).

   $T_n = 3n + 2$, $n = 1, 2, 3...$

   b) Sequence (neither arithmetic nor geometric) with bounded decreasing tendency (limit = 0). $T_n = \frac{1}{n^2}$, $n = 1, 2, 3...$

   c) Geometric sequence with unbounded increasing tendency (limit = ∞). $T_n = 3^n$, $n = 1, 2, 3...$

3. Let S_n denote sales in month n. Then $S_n = (2n - 1) + 0.5$

Sales in month 48 = S_{48} = (2 × 48 – 1) + 0.5 = 95.5 thousand = 95,500

Applications

Activity 1.9

Activity

Vary the discount rate

What will you do?

Using the information from Activity 1.8, do a sensitivity analysis by varying the discount rate to see whether the projects remain profitable under changed economic conditions.

Discuss the results.

Solutions

Let us assume that the inflation rate has increased and the required rate of return (the discount rate) increases to 10% p.a.

Project A: $NPV = -473.34$. Project A is no longer profitable.

Project B: $NPV = 157.78$. Project B is still profitable.

At a discount rate of 12% for Project B: $NPV = -815.87$. Project B is no longer profitable.

If the discount rate decreases below 8% p.a., each project will become more profitable.

Analysis of the profitability index *PI* will lead to the same conclusions.

Case study:
Decision on project investment

A company in Jamaica has 950,000 Jamaican dollars (JMD 950,000) to invest for four years. There are three mutually exclusive investment possibilities: A, B and C.

Investment A guarantees these cash inflows at the end of each of the four years:

Year	Investment A Cash inflows in JMD
1	300,000
2	400,000
3	500,000
4	600,000

Figure 1 Cash inflows for Investment A

Investment B is a four-year fixed investment which promises a simple interest rate of 12% p.a. The total value of the investment is paid out at the end of year four and there are no annual cash inflows at the end of years one to three.

Investment C is a treasury par bond with face value JMD 950,000, which gives the investor an annual cash inflow (coupon) of 6% p.a. The annual coupon is paid out at the end of each of the four years, and the initial investment or face value is paid back at the end of year four, together with the final coupon payment.

Activity 1.10

Activity

Decision on project investment

What will you do?

Analyse the possibilities from the case study and make a recommendation about which project the company should invest in.

Follow steps 1 to 5 to come to your decision.

1. Draw up a table for annual cash inflows for investments A, B and C.
2. Calculate the required rate of return (discount rate) for the three investments, assuming a tax rate of 30% p.a. on all investments and a return rate of 10% p.a. before tax.
3. Calculate the *NPV* and *PI* for the three investments.
4. Do a sensitivity analysis on the basis of possible future interest rate changes.
5. Write a report explaining your calculations and investment recommendation.

Solutions

1. At the end of year 0 (beginning of year 1), there is a cash outflow of 950,000 JMD. This can be expressed as an inflow of −950,000 and included in the table.

Year	Investment A Cash inflows in JMD	Investment B Cash inflows in JMD	Investment C Cash inflows in JMD
1	300,000	0	57,000
2	400,000	0	57,000
3	500,000	0	57,000
4	600,000	1,406,000	1,007,000

Investment C is a par bond, which means the price of the bond (initial value of investment) equals the face value of 950,000. The annual coupon is $0.06 \times 950{,}000 = 57{,}000$. At the end of year four, the face value is returned.

2. Rate of return after tax = 0.10 (1 – 0.30) = 0.07 = 7% (see "Discussion"). This is now used as discount rate for all investments.

3. Investment A: $NPV = 1{,}495{,}635.39 - 950{,}000 = 545{,}635.39$ (> 0)

   $PI = 1.574$ (> 1)

   Investment B: $NPV = 1{,}072{,}630.67 - 950{,}000 = 122{,}630.67$ (> 0)

   $PI = 1.129$ (>1)

Investment C: $NPV = 917{,}821.49 - 950{,}000 = -32{,}178.51$ (< 0)

$PI = 0.966$ (< 1)

Assuming all investment risks are equal and practically zero, Investment A is clearly the best. Investment C is not profitable since the coupon rate of 6% is lower than the required rate of 7%.

4. Various values for the discount rate can be applied. Because *NPV* and discount rates are inversely proportional, we always find *NPV* increasing with decreasing discount rates, and vice versa.
5. Students should present their analysis in the form of a report.

Activity 1.12

Activity

Calculate annuities

What will you do?

1. You pay a fixed amount of $50 per month at the end of each month for the next 10 years. The compound interest rate is 4% p.a. How much money will you have saved after 10 years?
2. What is the *PV* of annual payments of CAD 4,000 over five years at a rate of 3.4% p.a.?
3. A company in Pakistan wants to accumulate USD 10,000 over three years at an interest rate of 4% p.a. by depositing a fixed amount at the end of every month. Assume the exchange rate will stay fixed at USD 1 = PKR 80 (Pakistani rupees). What should the monthly amount be in PKR?
4. A loan of INR 50,000 must be amortised by paying equal monthly amounts of INR 2,000. The interest rate is 2.5% p.a. How long will it take to pay off the loan?

Solutions

1. The future value is $FV = R\dfrac{(1+i)^n - 1}{i}$

   The annual rate is converted to a monthly rate: $i = 0.04/12 = 0.0033$ and the period is $10 \times 12 = 120$ months.

   $$FV = 50\ \frac{(1+0.0033)^{120} - 1}{0.0033} = 7{,}346.98 \text{ dollars}$$

2. $$PV = R\frac{(1+i)^n - 1}{i(1+i)^n} = 4000\frac{(1+0.034)^5 - 1}{0.034(1+0.034)^5}$$

   $$= 18{,}111.47 \text{ CAD}$$

3. Calculate monthly amount *R* in USD. Values must be monthly values.

$$R = \frac{FV \times i}{(1+i)^n - 1}$$

$$= \frac{10000 \times 0.0033}{(1.0033)^{36} - 1} = 262.06 \text{ USD}$$

In PKR: $R = 262.06 \times 80 = 2{,}0964.96$

4. $$n = \frac{\ln R - \ln(R - iPV)}{\ln(1+i)}$$

$$= \frac{\ln 2000 - \ln(2000 - (0.025/12)50000))}{\ln(1 + 0.025/12)}$$

= 25.70 months = 2 years and 1.7 month

Unit 2

Linear equations and applications

Outcomes

Upon completion of this unit the students will be able to:

- **identify** linear relations;
- **interpret** the slope (gradient) and intercepts;
- **draw** graphs of linear functions;
- **solve** linear equations and inequalities;
- **solve** simultaneous linear equations; and
- **apply** linear equations to management situations.

Linear equations and straight-line graphs

These topics are very important for applications in supply and demand, budgeting, linear programming, linear regression, expected values, and so on. Although real-life situations probably mostly involve non-linear relations, such relations are often approximated by linear expressions.

Activity 1.15

Activity

Draw graphs

What will you do?

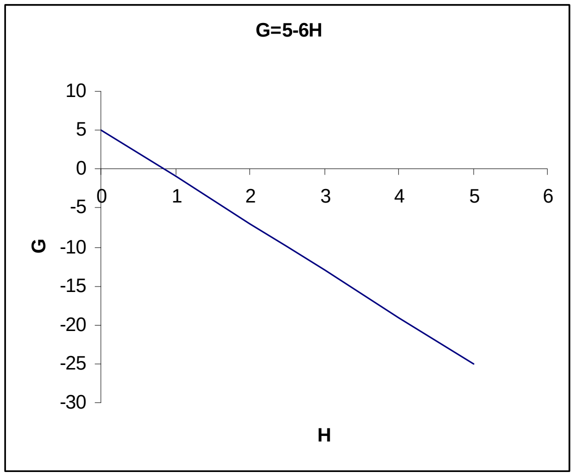
1. Draw the graph of the equation $G = 5 - 6H$ under the constraint $H \geq 0$.
2. Determine the slope and all intercepts of the line $y = 4x - 6$.
3. Draw the graph of $y = x + 3$.
4. Draw the graph and give the equation for the linear relationship between quantities x and y if you are given data points $(-2, -1)$ and $(3, 5)$. Take y as the dependent variable.
5. You are given the data points $(-1, 3)$, $(3, 6)$ and $(5, 9)$. Without plotting points, determine whether there is a linear relation between the points. Now plot the points and substantiate your answer. Discuss.
6. Write the following equations in standard form ($y = mx + c$):
   a) $4x + y - 9 = 0$
   b) $3y + 12x = 8$
7. Solve each of these inequalities by representing it as a shaded region in the (x, y)-plane:

a) $x - 2y \leq 3$

b) $3y + x + 1 \leq y - 6$

Solutions

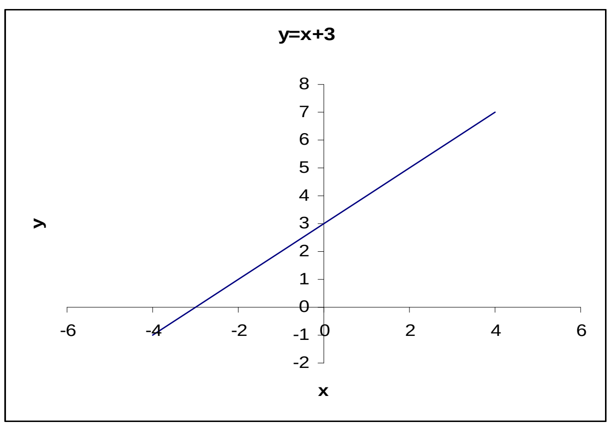
1.



2. Slope = 4. Intercept on x-axis is where $y = 0$: $4x - 6 = 0 \Rightarrow x = 1.5$

   Intercept on y-axis is where $x = 0$, $y = -6$

3.

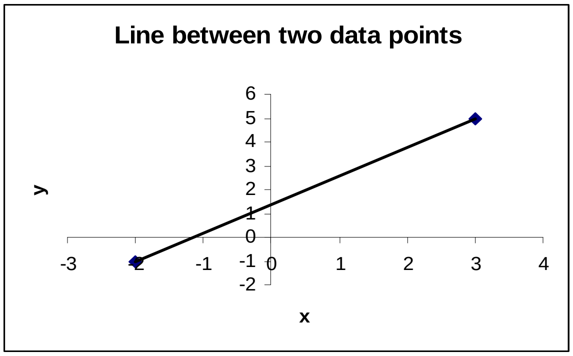


   The graph actually extends indefinitely in both directions. As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

4. The data points are at the extremes of the line. The line can be extended. The slope is: $m = \dfrac{5-(-1)}{3-(-2)} = 1.2$ and therefore

   $y = 1.2x + c$. Substitute either point, e.g. $5 = 1.2(3) + c$ gives $c = 1.4$

   Equations: $y = 1.2x + 1.4$



5. A single straight line can be drawn through any two given points in the plane. This is not necessarily true of any three given points. Determine the slope m_1 between (−1, 3) and (3, 6) and slope m_2 between (3, 6) and (5, 9).If the slopes are the same, the points lie on the same straight line and there is a linear relation. Now:

   $$m_1 = \frac{6-3}{3-(-1)} = 0.75$$

   $$m_2 = \frac{9-6}{5-3} = 1.5$$

   There is no linear relation. The graph will be piecewise linear.

6. a) $y = -4x + 9$

   b) $y = -4x + 8/3$

   The equations therefore represent parallel lines.

7. a) Write in the form $y \geq x/2 - 3/2$. Draw line $y = x/2 - 1.5$ with y-intercept at −1.5 and slope ½. Test for (0, 0) to satisfy inequality:
   $0 \geq 0/2 - 3/2$ is true. The region lies above the line and includes origin (0, 0).

   b) Write in the form $y \leq -x/2 - 7/2$. Draw line $y = -x/2 - 3.5$ with y-intercept at −3.5 and slope −½. Test for (0, 0) to satisfy inequality:
   $0 \leq 0/2 - 3.5$ is false. The region lies below the line and excludes origin (0, 0).

Simultaneous equations

Activity 1.16

Activity

Solve simultaneous equations

1. Solve these simultaneous equations both graphically and algebraically:

   a) $y = -x + 5, y = 5x - 7$

   b) $3y - x + 5 = 0, 2y = 2x - 1$

2. Discuss the possibility of solving these simultaneous equations:

   $y = -5x + 3.5, 2y + 10x + 6 = 0$

3. Discuss the possibility of solving these simultaneous equations:

   $3y = x + 9, -12y + 4x = -36$

4. Investigate whether it is possible to find a solution to these simultaneous equations:

   $y = -3x + 2, y = x - 4, 2y = -x + 5$

5. Summarise the results of questions 1– 4.

6. Describe, in terms of graphs, the implications of solving equations in three variables. You do not have to draw the actual graphs, but try to imagine them in space.

Solutions

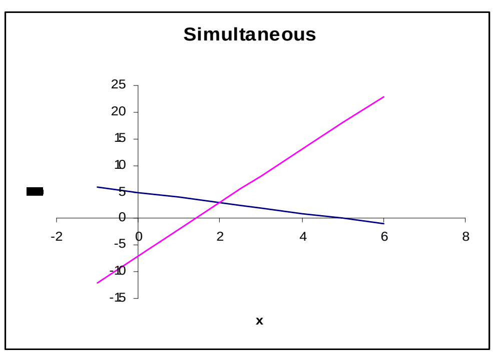
1. a) Algebraically: $-x + 5 = 5x - 7$

$$-6x = -12$$

$$x = 2$$

From $y = -x + 5$ we get $y = -2 + 5 = 3$

Graphically:



b) Algebraically: First, write in standard form.

$y = x/3 - 5/3$ and $y = 0.5x - 0.5$

Equate: $x/3 - 5/3 = 0.5x - 0.5$

$-x/6 = 7/6$

$x = -7$ and so $y = 0.5(-7) - 0.5 = -4$

Lecturers to review graph for correctness.

2. The equations are $y = -5x + 3.5$ and $y = -5x - 3$

   The slopes are the same and the lines therefore parallel. They do not intersect and there is no simultaneous solution.

3. The equations are $y = x/3 + 3$ and $y = x/3 + 3$

   The lines are identical. They do not intersect in a single point only but coincide everywhere. There are infinitely many solutions.

4. The lines intersect pair wise, but there is no point (x, y) satisfying all three equations.

5. Two straight lines either intersect in a unique point, or don't intersect (are parallel), or coincide. Two simultaneous linear equations in two variables, x and y, have a unique solution (x, y), no solution or infinitely many solutions. Three simultaneous linear equations in two variables have a unique solution only if two equations are identical and the third is different but not parallel.

6. Equations in three variables x, y and z represent planes in 3-D. Two planes intersect in a straight line (if they are not parallel or identical). Two equations in three variables therefore have infinitely many solutions, or no solution. However, three planes can intersect in a unique point (x, y, z).

Activity 1.17

Activity

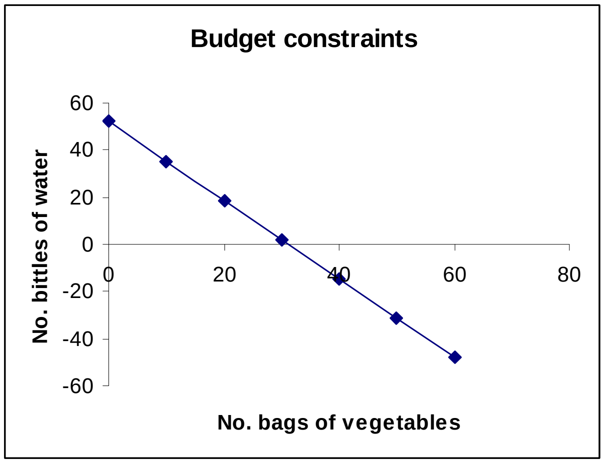
Budget constraint

What will you do?

1. Draw a graph for the budget constraint problem.
2. Solve the budget constraint problem for total budget $W = 10{,}000$ rupees.

Solutions

1. The model equation is $y = -1.67x + 52.08$, where x is the number of bags of vegetables and y the number of bottles of water.



It is now clear that we cannot buy more than 31 bags of vegetables. For $x = 40$, $y = -15$ approximately. This is impossible. We also cannot buy more than 52 bottles of water. The intercepts form the limits of what we can buy.

2. The budget constraint model is: $200x + 120y = 10{,}000$, which we re-write as $y = -1.67x + 83.33$. The analysis is similar to that in the Course Manual. If we buy no water, we can buy about 49 bags of vegetables. If we buy 30 bags of vegetables, we can buy $y = -1.67(30) + 83.33 = 33.23$, or 33 bottles of water.

Activity 1.18

Activity

Check the model

What will you do?

1. Interpret the results of the labour markets and wages model determined above.
2. Determine the equilibrium point graphically.

Solutions

1. Equilibrium or balance will be reached when the wages demanded by workers and wages offered by employers are equal. In this case, we see that 14 people can be employed. The corresponding wage per worker is 17 monetary units per period. One can also equate expressions for labour and solve for W first.

2.

Case study:
Break-even analysis

Firms that produce goods for sale have an income (revenue) from sales, but they also incur costs in the process. Generally, there are fixed costs and variable costs. Fixed costs are independent of the number of items produced, while variable costs are affected by the quantity of goods produced.

Assume that total costs equal fixed costs plus variable costs. We will also assume that variable costs are directly proportional to the number of items produced and nothing else, and that revenue comes only from selling items at a set price per unit.

Scenario: A small factory in India manufactures cell phone chargers and has determined that the price per unit will be 510.35 rupees (INR).

Fixed costs:	Rental	25,000
	Utilities	10,000
	Salaries	125,000
	Other	4,350
Variable costs:	Raw materials	0.6 INR × units produced
	Additional salaries	0.3 INR × units produced

Activity 1.19

Activity

Break-even report

What will you do?

Write a report on break-even for the factory that addresses these questions:

1. Denote total cost as C and number of items produced as Q. Derive the total cost equation $C(Q)$ for the factory.
2. Draw a graph of the total cost function.
3. Discuss the meaning of the values of the slope and intercept in this case.
4. Denote total income as I and derive the total income equation $I(Q)$ for the factory.
5. Graph the total income function.
6. Discuss the meaning of the values of the slope and intercept in this case.
7. Discuss and criticise the assumptions made to set up the model.
8. Define the meaning of the break-even point.
9. Calculate the break-even point algebraically and graphically.
10. Discuss your findings.

Solutions

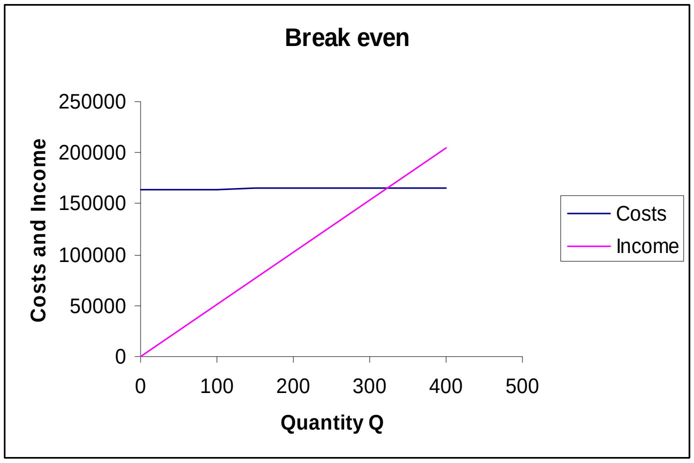
1. Total cost $C(Q)$ = variable costs + fixed costs

   $$= 0.9Q + 164{,}350$$

2. See 9.
3. Intercept on C-axis is 164,350. This means that even if no items are produced, there are still costs. The slope of 0.9 indicates that for every item produced, the cost incurred goes up by 0.9 INR.
4. Total income $I(Q) = 510.35Q$
5. See 9.
6. Intercept on I-axis is 0. This means that even if no items are produced, there is no income. The slope of 510.35 indicates that for every item produced, the income goes up by 510.35 INR.
7. The cost and income functions are very simple (linear) and do not take risk factors into account. We are also assuming number of units produced equals number of units sold.
8. Break-even is where costs and income are equal: $C(Q) = I(Q)$. At that stage, the company makes no profit but merely covers costs. It is important to know the value of Q at break-even so that we know how many units to produce (sell) to cover costs.
9. Algebraically: Set $C(Q) = I(Q)$, i.e. $0.9Q + 164{,}350 = 510.35Q$

This yields Q = 322.6 units. We should, therefore, produce at least 323 units to break even.

Graphically:



10. At 323 units, the income will be 164,843.05 INR. When we increase Q above 323, we see the Income line rising above the Costs line and a profit is made.

Unit 3

Non-linear functions and equations

Introduction

Quadratics, the exponential and logarithmic functions are important in statistics, business and finance, Probability theory and Utility theory.

Upon completion of this unit students will be able to:

Outcomes

- **solve** quadratic functions;
- **sketch** quadratic functions;
- **apply** quadratic functions in management problems; and
- **recognise** and use other non-linear functions (exponentials and logarithms).

Quadratics and their graphs

Activity 1.22

Activity

Draw some more

What will you do?

1. Consider the equation $y = 2x^2 + 3x - 2$
   a) Draw up a table of x and y values.
   b) Find the y-intercept.
   c) Calculate the value of Δ and interpret your answer.
   d) Draw the graph of the quadratic.
   e) Give the values of x and y where the quadratic reaches its maximum.

Solutions

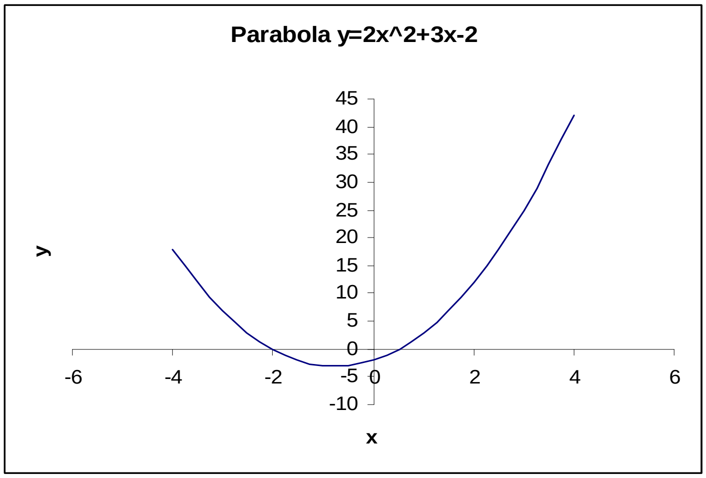
a)

x-values	y-values
−3	7
−2	0
−1	−3
0	−2
1	3
2	12
3	25

b) y-intercept = −2

c) $\Delta = b^2 - 4ac = 3^2 - 4(2)(-2) = 25 > 0$. The graph will intersect the x-axis in two places. These are the roots of the equation or the zeroes of the quadratic. The roots are: $x = \dfrac{-b-\sqrt{\Delta}}{2a}$ and $x = \dfrac{-b+\sqrt{\Delta}}{2a}$, i.e.
$x = -2$ and $x = 0.5$

d) The parabola extends indefinitely.



e) The quadratic does not reach a maximum. There is a minimum at $x = -0.75$ where $y = -3.125$. This is the turning point of the parabola.

Exponentials, logarithms and their graphs

Activity 1.23

Activity

Solve logarithms

1. Use a pocket calculator to determine the values of:
   a) e^3
   b) e^4
   c) e^7 and compare this to $e^3\,e^4$
2. Complete the table and discuss.

x-values	x^3-values	3^x values
0		
1		
2		
3		
4		

3. Solve for n if:
   a) $(1.1)^n = 40$
   b) $20\,(1 + 0.05)^n = 10$

Solutions

1. a) $e^3 = 20.0855$ (rounded off)
   b) $e^4 = 54.5982$ (rounded off)
   c) $e^7 = 1096.6332$ (rounded off) $= e^3 \times e^4$
2.

x-values	x^3-values	3^x values
0	0	1
1	1	3
2	8	9
3	27	27
4	64	81

The exponential function 3^x grows faster than the cubic x^3.

3. a) Apply the ln function: $\ln(1.1)^n = \ln 40$

$$n \ln 1.1 = \ln 40$$
$$n = \ln 40 / \ln 1.1$$
$$n = 38.7039$$

b) Apply the ln function: $\ln [20 (1 + 0.05)^n]= \ln 10$

$$\ln 20 + \ln(1 + 0.05)^n = \ln 10$$
$$n \ln(1.05) = \ln 10 - \ln 20$$
$$n = -14.21 \text{ (rounded)}$$

Apply non-linear functions

Case study:
Supply and demand

The supply and demand for a product manufactured in your factory is described by the following functions:

$P_S = 2Q_S^2 + Q_S - 4$ and $P_D = Q_D^2 - 2Q_D + 3$

Quantities are in hundreds and prices in thousands of units of your country's currency.

Separately for each function, determine the minimum price and quantities acceptable to buyers and to yourself.

Then determine the market equilibrium and discuss your findings.

Solution

Demand side: Minimum price P_D is where the parabola has a turning point at $Q_D = 1$. ($x=\dfrac{-b}{2a}$). Min $P_D = 2$. Therefore, a minimum demand of 1,000 at price 2 per unit.

Supply must be greater than 1,000 for a positive price. (Min P_S is at $Q_S = -1/4$ which, of course, is meaningless.)

Equilibrium: $2Q_S^2 + Q_S - 4 = Q_D^2 - 2Q_D + 3$ and $Q_S = Q_D = Q$

$$Q^2 + 3Q - 7 = 0$$

$$Q = \frac{-3-\sqrt{37}}{2} \text{ and } Q = \frac{-3+\sqrt{37}}{2}$$

Equilibrium quantity = 1.541 (rounded down) 1,541 units of the product. Equilibrium price = 2.29 monetary units per item.

If more units are produced and sold, the supply price goes up. This can occur when there is a greater demand for the product (perhaps created through marketing). At $Q = 3$ the supply price is $P_S = 17$ and the demand price is $P_D = 6$. However, good marketing can change the shape of the demand price function, say to $Q_D^2 - 2Q_D + 16$.

Unit 4

Differentiation and integration

Introduction

Calculus is the part of mathematics that studies changes in quantities. Rate of change is described by the process of differentiation, and integration is the "reverse" process of differentiation. Integration can also be seen as a process of summing and averaging.

Calculus is now done at secondary level in many schools and is no longer considered as "advanced" mathematics.

Upon completion of this unit students will be able to:

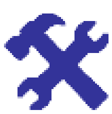
Outcomes

- **differentiate** simple functions;
- **calculate** rates of change of quantities;
- **perform** basic integration;
- **determine** maxima and minima of quantities; and
- **apply** differentiation and integration in management problems.

Differentiation

Differentiation is the tool for optimisation of quantities. We present the basics only. Lecturers may add to the notes if they wish.

Activity 1.26



Activity

Solve derivatives

What will you do?

1. Find the derivatives $\dfrac{dy}{dx}$ if:

   a) $y = 5x + 6$

   b) $y = -0.3x + 2$

   c) $y = 2x^2 + 4x + 3$

   d) $y = -0.8x^2 - x + 12$

   e) $y = 6x^3 - 0.5x^2 + 3x + 4$

2. Explain the meaning of the equation $\frac{dy}{dx} = -3$

   What relationship can be expected between y and x?

3. Explain the meaning of the equation $\frac{dy}{dx} = 2x + 4$

   What relationship can be expected between y and x? If the relationship is expressed graphically, what is the rate of change of y with respect to x at $x = 4$?

4. Quantities P and Q are related by the equation $2P = 6Q^2 - Q + 4$
   If $Q = 1$, what is the value of P? What is the rate of change of P with respect to Q at this point?

5. What is the slope of the line describing the relation
   $5x - 3y - 12 = 0$?

Solutions

1. a) $\frac{dy}{dx} = 5$ b) $\frac{dy}{dx} = -0.3$ c) $\frac{dy}{dx} = 4x + 4$

   d) $\frac{dy}{dx} = -1.6x - 1$ e) $\frac{dy}{dx} = 18x^2 - x + 3$

2. The slope of the graph of $y(x)$ is constant and negative. As x increases, y will decrease. For every unit increase in x, y decreases by 3 units. The relationship is linear.

3. The slope of the graph of $y(x)$ changes: it increases from negative for $x < -2$ to positive for $x > -2$. The slope is zero at $x = -2$, which indicates a turning point at $x = -2$. The function $y(x)$ must be a quadratic. In fact, $y = x^2 + 4x + k$ for some constant k.

   The rate of change at $x = 4$ is $\frac{dy}{dx} = 2x + 4$ (at $x = 4$) $= 12$

4. $P(Q) = 3Q^2 - 0.5Q + 2$, so $P(1) = 3(1)^2 - 0.5(1) + 2 = 4.5$

   $\frac{dP}{dQ} = 6Q - 0.5 = 5.5$ at $Q = 1$

5. Write: $5x - 3y - 12 = 0$ in the form $y = (5/3)x - 4$. Then $\frac{dy}{dx} = 5/3$

Optimisation: Maxima and minima

One of the most important applications of derivatives in decision science is to find optimal values of quantities.

Activity 1.28

Activity

Use more calculus methods

What will you do?

Use the methods of calculus to answer these questions:

1. Consider function $y = 4x - 2$ with constraint $x \geq -1$.
   a) Is the function increasing or decreasing? Justify your answer.
   b) Find the minima and maxima (if they exist).
   c) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
2. Consider function $y = -x^2 + 2x - 3$
   a) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
   b) Is the function increasing or decreasing? Justify your answer.
   c) Find the minima and maxima (if they exist).
   d) If you add the constraint $-2 \leq x \leq 1$, how does this change your answer for 2c)?
3. A certain function T has a turning point at $s = 6$.
   The value of $\frac{dT}{ds}$ is $\frac{dT}{ds} = s^2 - 36$
   Does the function have a minimum or maximum at $s = 6$?
4. If $\frac{dP}{dQ} = -2x + 5$, can you guess what type of relationship may exist between P and Q?

Solutions

1. a) Linear function with positive slope $m = 4$. Increasing from $x = -1$.
   b) There is no maximum. There is a minimum at $x = -1$: Minimum value is $y = -6$.
   c) $\frac{dy}{dx} = 4$ and $\frac{d^2y}{dx^2} = 0$
2. a) $\frac{dy}{dx} = -2x + 2$ and $\frac{d^2y}{dx^2} = -2$
   b) The function has the shape ∩ and increases for $x < 1$ where $\frac{dy}{dx} > 0$ and decreases for $x > 1$ where $\frac{dy}{dx} < 0$

c) $\frac{d^2y}{dx^2} < 0$ means there is a maximum where $\frac{dy}{dx} = 0$. Setting $-2x + 2 = 0$ yields $x = 1$. The maximum is at turning point (1, −2). Since the graph can be extended indefinitely, there is no minimum.

d) The constraint means the parabola is cut off at $x = -2$ and at $x = 2$. There are now two minima at $x = -2$ and at $x = 2$. The minimum value points are (−2, −11) and (2, −3).

3. Calculate $\frac{d^2T}{ds^2} = \frac{d}{ds}[s^2 - 36] = 2s$. At $s = 6$ we have $\frac{d^2T}{ds^2} = 12 > 0$ and so T has a minimum at $s = 6$.

4. Quadratic.

Integration

Integration can, at a simplified level, be seen as the reverse operation to differentiation. It is used extensively in probability density functions for continuous random variables such as the normal distribution.

Indefinite integrals

$$\int \frac{dy}{dx}\,dx = y(x) + K$$

Activity 1.30

Activity

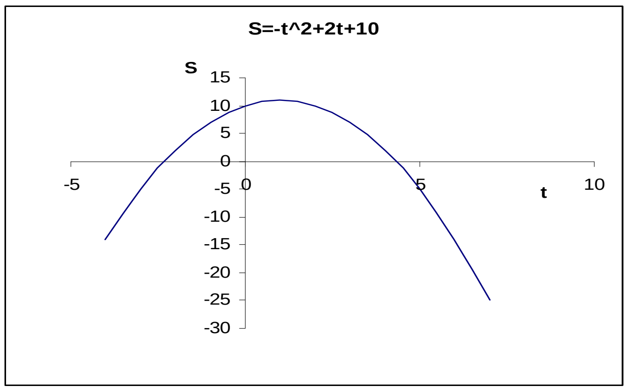
Calculate the integrals

What will you do?

1. $\int (2x - 2)dx$
2. $\int (-x + 1)dx$
3. $\int (4 - 4x)dx$
4. $\int (x^2 + 3x - 1)dx$
5. Find $P(Q)$ if $\frac{dP}{dQ} = 5$ and it is given that the graph has P-intercept equal to 4. Draw the graph of P.
6. Find $S(t)$ if $\frac{dS}{dt} = -2t + 2$ and it is given that the graph has S-intercept equal to 10. Draw the graph and determine the maximum value of S.

Solutions

1. $x^2 - 2x + K$
2. $-\frac{1}{2}x^2 + x + K$
3. $4x - 2x^2 + K$
4. $\frac{1}{3}x^3 + (3/2)x^2 - x + K$
5. $P(Q) = 5Q + K$. We are given $P(0) = 4$, but $P(0) = K$ and so $K = 4$. Therefore $P(Q) = 5Q + 4$
6. $S(t) = -t^2+2t + K$ and $S(0) = 10$. Therefore $S(t) = -t^2+2t + 10$. This is a parabola with shape ∩. The maximum is where $\frac{dS}{dt} = -2t + 2 = 0$. This is where $t = 1$. Maximum value is $S(1) = 11$.



Definite integrals

Definite integrals have the form $\int_a^b y\,dx$. The numbers a and b are the bounds for the integral.

$$\int_a^b \frac{dy}{dx}\,dx = y(x)\Big|_{x=a}^{x=b} = y(b) - y(a)$$

Activity 1.31

Activity

Calculate and interpret

What will you do?

Calculate $\int_{-1}^{1} x\,dx$ and interpret your answer.

Solution

$$\int_{-1}^{1} xdx = ½\, x^2 \Big|_{x=-1}^{x=1} = ½\,(1)^2 - ½\,(-1)^2 = 0$$

The area of the triangular region between the graph of $y = x$ and x-axis between $x = -1$ and $x = 0$ is 0.5 and equals the area of the triangle below the x-axis between $x = 0$ and $x = 1$. The "net total area" between the graph of $y = x$ and the x-axis between $x = -1$ and $x = 1$ is thus zero!

Assessment

Assessment — Module 1

1. If the amount \$10,500.00 includes tax of 12%, what was the original amount excluding tax?
2. What is the rate of return per annum on an investment with value that increases from 12,000 to 13,634 over the year?
3. You are given the sequence 2, 3, 4.5, 6.75…
   a) Find the formula for the general term of the sequence.
   b) Determine the sum of the first 20 terms of the series.
4. An investment of 10,000 cedi has a cash inflow of 4,500 cedi at the end of each year for the next three years. The discount rate is 4.5%.
   a) Calculate the present value of all cash flows.
   b) Determine the *NPV* and *PI* for the investment. Discuss your findings.
   c) Do a sensitivity analysis for a scenario where you expect the interest rate to go up.
5. Draw the graph and give the equation for the linear relationship between quantities P and Q if you are given the two data points (P, Q): (22, 4) and (46, 12). Take Q as the dependent variable.
6. Solve these simultaneous equations both graphically and algebraically:

   $2y - 2x + 6 = 0$ and $2y = x - 5$

   Now simultaneously solve the inequalities $2y - 2x + 6 \geq 0$ and $2y \geq x - 5$
7. A cell phone stall allows customers to make calls of three minutes' duration. Let Q denote the average number of phone calls made per week. Costs are denoted in units of dollars.

   The total weekly cost function for supplying the service is $C = 3Q + 50$

   The total weekly income function for this service is $I = 100Q - 0.5Q^2$

   Analyse the situation for break-even and profit, following the method in Unit 4.
8. Find the maximum value of $f(x)$ if $f(x) = -3x^2 + 5x - 1$
9. Determine the value of the definite integral and interpret it:
   $\int_{-2}^{2}(-x^2 + 4)dx$
10. Let the total cost of production be y and let x denote the number of items manufactured. The marginal cost is $\text{MC} = \frac{dy}{dx}$

A small factory in Malaysia has produced 200 items. It would like to expand its operations and produce 300 items, but only has 20,000 rupees available to fund expansion. It is known that marginal cost MC = $\frac{dy}{dx} = x - 4$ at this stage. What would be the total cost for the next 100 items to be manufactured? Can the factory afford it?

Solutions

1. If original amount is X, then $X(1+0.12) = 10{,}500.00$ and $X = 9{,}375.00$.

2. Let the rate be R. Then $12{,}000(1 + R) = 13{,}634$

$$(1 + R) = 1.1362$$

$$R = 0.1362 = 13.62\%$$

3 a) $T_n = 2(1.5)^{n-1}$, n=1, 2, 3...

b) This is a geometric sequence with ratio $r = 1.5$, so:

$$S_n = \frac{T_1(1-r^n)}{1-r} = \frac{T_1(r^n-1)}{r-1}$$

$$S_{20} = \frac{2(1.5^{20}-1)}{1.5-1} = 13{,}297.03$$

4. a) $$PV = \frac{4500}{(1+0.045)} + \frac{4500}{(1+0.045)^2} + \frac{4500}{(1+0.045)^3}$$

$= 12{,}370.34$ cedi

b) $NPV = PV - 10{,}000 = 2{,}370.34$ cedi
$PI = 1.237$
The investment is profitable ($NPV > 0$, $PI > 1$) as long as there is no risk.

c) Assume the discount rate goes up to 8%. Then:

$$NPV = \frac{4500}{(1+0.08)} + \frac{4500}{(1+0.08)^2} + \frac{4500}{(1+0.08)^3} - 10{,}000$$

$= 11{,}596.94 - 10{,}000 = 1{,}596.94$ (> 0)

The investment is still profitable (although less so). Even at 12% $NPV > 0$. The investment seems quite stable under changes in interest rate.

5. $m = \frac{8}{24} = \frac{1}{3}$ so $Q = \frac{1}{3}P + c$. Substitute (22, 4): $4 = \frac{22}{3} + c$

Linear relation: $Q = \frac{1}{3}P - \frac{10}{3}$

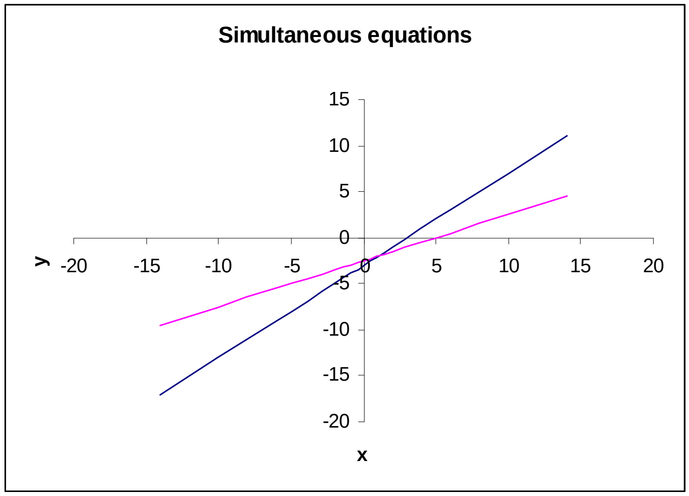
The graph has slope $\frac{1}{3}$ and Q-intercept $-\frac{10}{3}$

Lecturers should review graph for correctness.

6. Rewrite equations: $y = x - 3$ $y = 0.5x - 2.5$

$x - 3 = 0.5x - 2.5$ implies $0.5x = 0.5$ and $x = 1$

Point of intersection: $(1, -2)$



(0, 0) lies in both regions $2y - 2x + 6 \geq 0$ and $2y \geq x - 5$. The region described by the inequalities simultaneously is the intersection of the areas above the two lines.

7. Break-even: $C = 3Q + 50$ must equal $I = 100Q - 0.5Q^2$

$3Q + 50 = 100Q - 0.5Q^2$

This can be written as: $Q^2 - 194Q + 100 = 0$

$$= \frac{-b - \sqrt{\Delta}}{2a} \text{ or } Q = \frac{-b + \sqrt{\Delta}}{2a}$$

$Q = -200$ or $Q = 193.48$

At 193 calls per week, they will have break-even. To make a profit, they need customers to make more than 194 calls per week.

8. Maximum is where $\dfrac{df}{dx} = -6x + 5 = 0$; i.e. $x = 5/6$

Max $f(x) = -3(5/6)^2 + 5(5/6) - 1 = 1.083$

9. $$\int_{-2}^{2} (-x^2 + 4)dx$$

$$= [-\tfrac{1}{3}x^3 + 4x\,]\Big|_{x=-2}^{x=2} = [-\tfrac{1}{3}(2)^3 + 4(2)] - [-\tfrac{1}{3}(-2)^3 + 4(-2)]$$

$$= 16/3 - [-16/3] = 32/3$$

It is the total area between the graph of $(-x^2 + 4)$ and the x-axis between $x = -2$ and $x = 2$.

10. $\text{MC} = \dfrac{dy}{dx} = x - 4$

Additional cost = $\int_{200}^{300} (x-4)dx$

$= [\tfrac{1}{2}x^2 - 4x\,]\Big|_{x=200}^{x=300} = [\tfrac{1}{2}(300)^2 - 4(300)] - [\tfrac{1}{2}(200)^2 - 4(200)]$

= 24,600 rupees

The cost of expansion is slightly more than the factory can afford.