Equations 5

The aim of this unit is to equip the learners with the concept of equations. The principal foci of this unit are degree of an equation, inequalities, quadratic equations, simultaneous linear equations, graphical equation and their applications in solving business problems followed by examples.

Blank Page

Lesson 1: Equation and Identity

After studying this lesson, you should be able to:

- Explain the nature and characteristics of equations;
- Explain the nature and characteristics of identities;
- Solve the equations;
- Solve the inequalities.

Introduction

Many applications of mathematics involve solving equation. In this lesson we will discuss the equation, identities and uses of equations.

Equation

An equation is a statement which says that two quantities are equal to each other. An equation consists of two expressions with a '=' sign between them. In other words, if two sides of an equality are equal only for particular value of the unknown quantity or quantities involved, then the equality is called an equation.

An equation is a statement which says that two quantities are equal to each other.

For example, $4x = 8$ is true only for $x = 2$. Hence, it is an equation.

An equation which does not contain any variable is either a true statement, such as 2 + 3 = 5, or a false statement, such as 3 + 5 = 12. If an equation contains a variable, the solution set of the equation is the set of those values for the variable which gives a true statement when substituted into the equation.

For example, the solution set of $y^2 = 4$ is (–2, 2), because $(-2)^2 = 4$, and $2^2 = 4$, but $y^2 \neq 4$ if y is any number other than –2 or 2.

Identities

The equations signify relation between two algebraic expressions symbolized by the sign of equality. If two sides of an equality are equal for all values of the unknown quantity or quantities involved, then the equality is called an identity.

If two sides of an equality are equal for all values of the unknown quantity or quantities involved, then the equality is called an identity.

For example, $x^2 - y^2 = (x + y)(x - y)$ is an identity.

We can prove that identities hold true for whatever are values of the variables substituted in these. It we use x = 2 and y = 3 in the above identity, we have $(2)^2 - (3)^2 (2 + 3)(2 - 3)$

or, $4 - 9 (5)(-1)$

or, $-5 = -5$

Again, by substituting the values of x = – 4 and y = – 6, we have

$(-4)^2 - (-6)^2 = (-4 - 6)(-4 + 6)$

or, $16 - 36 = (-10)(2)$

or, $-20 = -20$

Hence, identities hold true for whatever value is put for variables.

Derived Identities

Derived identities are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics.

Derived identities are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics. For example,

(1) Identity → $(x + y)^2 = x^2 + 2xy + y^2$ .... (i)
Derived Identities $x^2 + y^2 = (x + y)^2 – 2xy$
and $2xy = (x + y)^2 – (x^2 + y^2)$

(2) Identity → $(x – y)^2 = x^2 – 2xy – y^2$ .... (ii)
Derived Identities $x^2 + y^2 = (x – y)^2 + 2xy$
and $2xy = x^2 + y^2 – (x – y)^2$

By adding (i) and (ii)

$(x + y)^2 + (x –y)^2 = 2(x^2 + y^2)$ .... (iii)

By substituting (ii) from (i), we get

$(x + y)^2 – (x – y)^2 = 4xy$ .... (iv)

By dividing both (i) and (ii) by 4 and then subtracting (ii) from (i), we have $[(x + y)^2 / 4] – [(x – y)^2]$

The following section of this lesson contains some model applications of equations.

Example–1:

Solve, $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \dfrac{1}{3}$

Solution:

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{1}{3}$$

By cross multiplying, we have

$$3\sqrt{1+x}-3\sqrt{1-x} = \sqrt{1+x}+\sqrt{1-x}$$

or, $3\sqrt{1+x}-3\sqrt{1+x} = \sqrt{1-x}+\sqrt{1-x}$

or, $2\sqrt{1+x} = 4\sqrt{1-x}$

Squaring both sides, we have,

$4(1 + x) = 16 (1 – x)$

or, $4 + 4x = 16 – 16x$

Transposing the term 16x and 4

$4x + 16x = 16 – 4$

or, $20x = 12$

or, $x = \frac{3}{5}$

Therefore, $x = \frac{3}{5}$ is the solution of the given equation.

Example – 2:

The sum of two numbers is 45 and their ratio is 7:8. Find the numbers.

Solution:

Let, one of the numbers be x

The other number is (45 – x)

Using the given information, we get, $\frac{x}{45-x} = \frac{7}{8}$

By cross multiplication, we get

$8x = 7(45 - x)$

or, $8x = 315 - 7x$

Transposing the term –7x, we have

$8x + 7x = 315$

or, $15x = 315$

or, $x = \frac{315}{15} = 21$

Hence, the one number is 21 and the other number is (45 – 21) = 24.

Example – 3:

The ages of a mother and a daughter are 31 and 7 years respectively. In how many years will the mother's age be $\frac{3}{2}$ times that of the daughter?

Solution:

Let the required number of years be *x*.

Mother's age after *x* years = 31 + x

Daughter's age after *x* years = 7 + x

Using the given information, we get

$31 + x = \frac{3}{2}(7 + x)$

or, $31 + x = \frac{21 + 3x}{2}$

By cross multiplication, we have

$2(31 + x) = 21 + 3x$

or, $62 + 2x = 21 + 3x$

Transposing the terms 3x and 62

$$2x - 3x = 21 - 62$$

or, $-x = -41$

or, $x = 41$

Hence, in 41 years, the mother's age will be $\frac{3}{2}$ times age of the daughter's.

Example–4:

The distance between two stations is 340 km. Two trains start at the same time from these two stations on parallel tracks to cross each other. The speed of one train is greater than that of other by 5 km / hr. If the distance between the two trains after 2 hours of their start is 30 km, find the speed of each trains?

Solution:

Let the speed of the first train be x km / hr.

Then the speed of the second train be $(x + 5)$ km / hr.

Distance covered by the first train in 2 hrs = $2x$ km.

Distance covered by the second train in 2 hrs = $2(x + 5)$ km = $(2x + 10)$ km.

Since, both the trains are in opposite directions.

Total distance between the two stations = 340 km.

Distance between the two trains after 2 hours = 30 km.

Therefore, distance covered by two trains in two hours from opposite directions

$$= (340 - 30) = 310 \text{ km}$$

$$\therefore 2x + (2x + 10) = 310$$

or, $4x = 300$

$$x = 75$$

Hence, speed of the first train = 75 km / hr and the speed of the second train = 80 km / hr.

Questions For Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is an equation and inequalities? Mention the characteristics of equation.
2. What is the difference between identity and equation? Give examples.
3. Solve the following equations:

   (i) x (x + 1) + 72 / x (x+1) = 18

   (ii) $x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$

   (iii) $\sqrt{\dfrac{x}{x+6}} + \sqrt{\dfrac{x+6}{x}} = \dfrac{25}{12}$
4. Wasifa's mother is four times as old as Wasifa. After five years, her mother will be three times as old as she will be then, what are their present ages?
5. A steamer goes downstream and covers the distance between two parties in 4 hours while it covers the same distance upstream in 5 hours. If the speed of the stream is 21 cm/per hour, find the speed of the steamer in still water.
6. Three prizes are to be distributed in a quiz contest. The value of the 2nd prize is five-sixths the value of the first prize and the value of the third prize is four fifths that of the second prize. If the total value of the three prizes is Tk.15,000, find the value of each prize.
7. The price of two cows and five horses is Tk.68,000. If the price of horse exceeds that of a cow by Tk.800, find the price of each.

Multiple choice questions (√ the appropriate answers)

1. The difference between two numbers is 9 and the difference between the squres is 207. The numbers are:

   a) 17, 8 b) 16, 7 c) 23, 14
2. The ratio of two numbers is 3 : 5. If each is increased by 10 the ratio becomes 5 : 7, the numbers are:

   a) 6, 10 b) 15, 25 c) 9, 15
3. The sum of three numbers is 102. If the ratio between the first and the second be 2 : 3 and that of between the second and the third be 5 : 3, the second number is

   a) 30 b) 45 c) 27
4. Taka 49 were divided among 150 children. Each girl got 50 paisa and each boy got 25 paisa. How much boys were there?

   a) 104 b) 105 c) 102
5. If $2x + 3x = 5$ and $x = -2$, then the value of y is:

   a) 1/3 b) 1 c) 3

Lesson-2: Inequality

After studying this lesson, you should be able to:

- Describe the nature of inequalities;
- Explain the properties of inequality;
- Solve the inequalities.

Nature of Inequality

Relationship of two expressions with an inequality sign between them is called inequality.

Relationship of two expressions with an inequality sign (≤ or ≥, < or >) between them is called inequality. For example,

$x > y \rightarrow$ "x is greater than y"

$x < y \rightarrow$ "x is smaller than y"

$x \ngtr y \rightarrow$ "x is not greater than y"

$x \nless y \rightarrow$ "x is not smaller than y"

$x \leq y \rightarrow$ "x is smaller than or equal to y"

$x \geq y \rightarrow$ "x is greater than or equal to y"

Properties of Inequalities

The fundamental properties of inequalities are as follows:

(a) Order Axioms: If x and b are only elements, then

(i) One and only one of the following is true:

$x = b$, $x < y$ and $x > y$

(ii) If $x < y$ and $y < z$, then $y < c$

(iii) If $x < y$ and $x < z$, then $xz < yz$

Since, '$x > y$' and '$y < x$' are the same statements, the above axioms can be replaced in terms of '$x > y$'.

As shown earlier sometimes equality signs are combined with inequality signs $x < y$ means $x = y$ or $x < y$.

Again $x < y$ means x is not less than y and that means either $x > y$.

So, $x < y$ means $y \leq x$.

We also say that x is positive when $x \geq 0$ and x is negative, when $x < 0$.

(b) Operation Axioms:

All equals may be add or subtracted from both sides of inequalities and the inequality is preserved.

(i) All equals may be add or subtracted from both sides of inequalities and the inequality is preserved.

For example, if $5x - 9 < 12$

We may add 5 to both sides and we get

$5x - 9 + 5 < 12 + 5$

or, $5x - 4 < 17$

Any term in an inequality can be moved from one side to the other provided that its sign is changed. For example, if $5x - 4 < 17$.

or, $5x < 17 + 4$

And, again if $x - z > y$

or, $x > y + z$

(ii) Both sides of an inequality may be multiplied or divided by a positive number and the inequality is preserved. For example, if $12x < 36$.

Both sides of an inequality may be multiplied or divided by a positive number and the inequality is preserved.

After Multiplying the both sides of inequality by 5, we get

$12x \times 5 < 36 \times 5$

or, $60x < 180$

Again, after dividing both sides of inequality by 3, we get

$12x \div 3 < 36 \div 3$

or, $4x < 12$

(iii) Both sides on an inequality are multiplied or divided by a negative number and the direction of the inequality is reversed. For example, if $7x < 40$, (where $x = 4$).

Both sides on an inequality are multiplied or divided by a negative number and the direction of the inequality is reversed.

By multiplying both sides of the inequality by –5, we have

$7x \times (-5) > 40 \times (-5)$ [Note that inequality sign has been changed from < to >]

or, $-35x > -200$

This is because when $x = 4$, the inequality $-35x = -140$ is greater than –200.

(iv) An inequality can by converted into an equation:

If $x > y$ then $x = y + p$

Where p is the positive real number (i.e. $p > 0$)

If $z > m$, then we write

$z = m + q$, where $q > 0$

Hence, $x, z = (y + p)\ (m + q) = ym + yq + pm + pq$

Now p and q are positive. If in addition y and m are positive, then every term on the right-hand side is also positive so that

$x.\ z > y.\ m$

(v) If $\frac{x}{z} > \frac{y}{m}$, then $\frac{z}{x} < \frac{m}{y}$

(vi) If $x < y$, then $-x > -y$

(vii) Now if $x_1 > y_1,\ x_2 > y_2,\ x_3 > y_3$ ----------- $x_n > y_n$,

then $x_1 + x_2 + x_3 +$ ----------- $+x_n > y_1 + y_2 + y_3 +$ --------- $+y_n$

and $x_1.\ x_2.\ x_3$ --------- $x_n > y_1.\ y_2.\ y_3$ ----------- y_n

(viii) If $x > y$ and $n > 0$ then $x^n > y^n$ and $\frac{1}{x^n} < \frac{1}{y^n}$

The following section of this lesson contains some applications of inequality.

Example-1:

Solve: $3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$

Solution:

$3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$

or, $3[4x - 10x - 15)] \leq 7 - 2\ [x + 12 - 3x)]$

or, $12x - 30x - 45 \leq 7 - 2x + 24 + 6x$

Transposing both sides we have

or, $12x - 30x + 2x \leq 6x - 24 - 45$

or, $-12x \leq -62$

Multiplying both sides by –1, we get

or, $22x \geq 62$

or, $x \geq \frac{31}{11}$

Example-2:

Solve: $5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$

Solution:

$5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$

or, $5x - 6x - 8) > 8x - 12 + 36x$

or, $5x - 6x - 8x - 36x > -12 - 8$

or, $-45x > -20$,

Multiplying both side we get –1,

or, $45x < 20$

or, $x < \frac{20}{45}$

i.e. $x < \frac{4}{9}$

Questions for review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Solve the following inequalities:

   (i) $3x - 2 < 4 + 6x$

   (ii) $2x - 3(4 - x) < 7 - 4(1 - 2x)$

   (iii) $2x - 3 + 4(5 - 3x) \geq 4x - 19$

   (iv) $3 - [5x + 11 - 2x(3 + 2)] \leq 11 - 3x[x - 5(3 - x)]$

   (v) $[(5x - 7) / (2x - 3)] - 2(3 - 4x) < 0$

2. Solve each inequality with a sign graph

   (i) $(x + 3)(x + 4)^2 < 0$

   (ii) $(x - 1)(x - 2)(x - 3) < 0$

Multiple choice questions (√ the appropriate answers)

1. If $x + 2y \leq 3$, $x > 0$ and $y > 0$, then one of the solution is

   a) $x = -1, y = 2$ b) $x = 1, y = 1$ c) $x = 2, y = 1$

2. The system of linear in equalities $x + y \leq 0$, $x \geq 0$ and $y \geq 0$, has

   a) exactly 1 solution b) 3 solutions c) no solution

3. The value of x from linear inequality $4x - 7 > 6x + 5$

   a) $x < -6$ b) $x < -8$ c) $x < -4$

Lesson-3: Degree of an Equation

After studying this lesson, you should be able to:

- Explain the nature of degree of an equation;
- Solve the simultaneous linear equation.

Nature of Degree of an Equation

An ordinary equation involving only the first power of the unknown quantity is called 'simple' or 'linear' equation or equation of the first degree.

An equation involving only one unknown quantity is called ordinary equation. An ordinary equation involving only the first power of the unknown quantity is called 'simple' or 'linear' equation or equation of the first degree. When the highest power of the unknown quantity x is 2, it is called 'quadratic or the second degree equation; when the highest power of the unknown quantity x is 3, the equation is termed as 'cubic' or the third degree equation. When the highest power of x is 4, the equation is called 'biquadratic or the fourth degree equation.

For example,

$2x + 18 = y \rightarrow$ Linear equation
$2x^2 + 5x + 7 = 0 \rightarrow$ Quadratic equation
$x^3 + 5x^2 + 3x + 9 = 0 \rightarrow$ Cubic equation
$x^4 + 10x^3 + 5x^2 + 2x + 10 = 35 \rightarrow$ Biquadratic equation

If an equation in x is unaltered by changing x to $\frac{1}{x}$, it is known as a reciprocal equation.

An equation in which the variable occurs as indices or exponents is called an exponential equation.

For example, $3^x = 21$, $81^x = 9^{x+4}$ etc. are called exponential equations.

If more than one unknown quantity are involved, the number in independent equations required for solution is equal to the number of the unknown quantities. Such set of linear equations is called simultaneous linear equations.

The following section of this lesson contains some model applications.

Example-1:

Solve $4x^4 - 16x^3 + 23x^2 - 16x + 4 = 0$

Solution:

$4x^4 - 16x^3 + 23x^2 - 16x + 4 = 0$

Rearranging the terms, we have

$4x^4 - 4 - 16x^3 - 16x + 23x^2 = 0$

Dividing both sides by x^2 we have,

$$4x^2 + \frac{4}{x^2} - 16x - \frac{16}{x} + 23 = 0$$

or, $4\left(x^2+\dfrac{1}{x^2}\right)-16\left(x+\dfrac{1}{x}\right)+23=0$

or, $4\left[\left(x+\dfrac{1}{x}\right)^2-2.x.\dfrac{1}{x}\right]-16\left(x+\dfrac{1}{x}\right)+23=0$

Putting y for $x+\dfrac{1}{x}$, we have

$4(y^2-2)-16(y)+23=0$

or, $4y^2-8-16y+23=0$

or, $4y^2-16y+15=0$

or, $4y^2-10y-6y+15=0$

or, $2y(2y-5)-3(2y-5)=0$

or, $(2y-5)(2y-3)=0$

either, $2y-5=0$ or, $2y-3=0$

or, $2y=5$ or, $2y=3$

or, $y=\dfrac{5}{2}$ or, $y=\dfrac{3}{2}$

When, $y=\dfrac{5}{2}$, then $x+\dfrac{1}{x}=\dfrac{5}{2}$

or, $\dfrac{x^2+1}{x}=\dfrac{5}{2}$

or, $2x^2+2=5x$

or, $2x^2-4x-x+2=0$

or, $2x(x-2)-1(x-2)=0$

or, $(2x-1)(x-2)=0$

either, $2x-1=0$ or, $x-2=0$

or, $2x=1$ $\therefore x=2$

$\therefore x=\dfrac{1}{2}$

When, $y=\dfrac{3}{2}$, then $x+\dfrac{1}{x}=\dfrac{3}{2}$

$\dfrac{x^2+1}{x}=\dfrac{3}{2}$

or, $2x^2+2=3x$

or, $2x^2-3x+2=0$

We know that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (Here, a = 2, b = –3, c = 2)

$$x = \frac{3 \pm \sqrt{9-16}}{4} = \frac{3 \pm \sqrt{-7}}{4} = \frac{3 \pm \sqrt{7}i}{4} \quad \left(Here,\ i = \sqrt{-1}\right)$$

Hence, $x = 2,\ \frac{1}{2}$ or, $\frac{3 \pm \sqrt{7}i}{4}$

Example-2:

Solve $x + \frac{4}{y} = 1$

$$y + \frac{4}{x} = 25$$

Solution:

$$x + \frac{4}{y} = 1 \quad \text{.......................................} \quad (i)$$

$$y + \frac{4}{x} = 25 \quad \text{....................................} \quad (ii)$$

From equation (i) $x + \frac{4}{y} = 1$

or, $\frac{xy + 4}{y} = 1$

or, $xy + 4 = y$ .... (iii)

From equation (ii) $y + \frac{4}{x} = 25$

or, $\frac{xy + 4}{x} = 25$

or, $xy + 4 = 25x$ .... (iv)

Subtracting the equation (iii) from equation (iv)

$0 = 25x - y$

or, $y = 25x$

Putting the value of y in equation (iii), we get

$x\,(25x) + 4 = 25x$

or, $25x^2 + 4 = 25x$

or, $25x^2 - 25x + 4 = 0$

or, $25x^2 - 20x - 5x + 4 = 0$

or, $5x\,(5x - 4) - 1\,(5x - 4) = 0$

or, $(5x - 4)\,(5x - 1) = 0$

either, $5x - 4 = 0$ or, $5x - 1 = 0$

or, $5x = 4$ or, $5x = 1$

so, $x = \frac{4}{5}$ so, $x = \frac{1}{5}$

Now $x = \frac{4}{5}$, then $y = 25 \times \frac{4}{5} = 20$

$x = \frac{1}{5}$, then $y = 25 \times \frac{1}{5} = 5$

Thus, $x = \frac{4}{5}$, $x = \frac{1}{5}$

$y = 20$, $y = 5$

Example-3:

Find the solution of the system of equations.

$4x - 3y + z = 1$ .... (1)

$2x - y + 2z = 6$ .... (2)

$3x + 4y - 4z = -1$ .... (3)

Solution:

Let us first eliminate z:

We rewrite: $2 \times (1)$: $8x - 6y + 2z = 2$

$\underline{(2): 2x - y + 2z = 6}$

Subtraction: $6x - 5y = -4$ .... (4)

Now we rewrite: (3) : $3x + 4y - 4z = -1$

$\underline{2 \times (2): 4x - 2y + 4z = 12}$

Addition: $7x + 2y = 11$ .... (5)

Let us now eliminate y from (4) and (5); i.e.

$5 \times (5)$: $35x - 10y = 55$

$\underline{2 \times (4): 12x - 10y = -8}$

Addition: $47x = 47$

So, $x = 1$.

Now, substituting the value of x in (5), we get the value of y, i.e.

$7(1) + 2y = 11$

or, $2y = 11 - 7$

or, $2y = 4$

so, $y = 2$

Now, substituting the value of x and y in (1), we have:

$4(1) - 3(2) + z = 1$

or, $z = 1 - 4 + 6$

or, $z = 3$

Therefore the solution is: $x = 1$, $y = 2$ and $z = 3$.

Questions for Review:

These questions are designed to help you assess how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. Explain the nature of degree of an equation.
2. Solve the following equations:

   (i) $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$

   (ii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{3}{2}$

   $x - y = 3$

   (iii) $3x + 7 - 5z = 0$

   $7x - 3y - 9z = 0$

   $x^2 + 2y^2 + 3z^2 = 23$

3. Solve

   (i) $x^3 + y^3 = 4914$

   $x + y = 18$

   (ii) $27^x = a^y$

   $81^y = 243.3^x$

Multiple choice questions (√ the appropriate answers)

1. The solution of the simultaneous linear equation $\frac{2x}{3} - \frac{y}{2} = -\frac{1}{6}$ and $\frac{x}{2} - \frac{2y}{3} = 3$

   a) $x = 2,\ y = 3$ b) $x = -2,\ y = 3$ c) $x = 2,\ y = -3$

2. The solution of the system of simultaneous linear equation $4x - 3y = 7$ and $7x + 5y = 2$ is:

   a) $x = -1,\ y = 1$ b) $x = 1,\ y = -1$ c) $x = 1,\ y = 1$

3. If $7x + 9y = 85$ and $4x + 5y = 48$, then

   a) $x = 7,\ y = 4$ b) $x = \frac{40}{7},\ y = 5$ c) $x = 37,\ y = 8$

4. The solution of the pair of equations $\frac{3}{x} + \frac{6}{y} = 12$ and $\frac{4}{x} + \frac{12}{y} = 44$ is:

   a) $x = \frac{1}{-2},\ y = 3$ b) $x = -2,\ y = 3$ c) $x = -\frac{1}{2},\ y = \frac{1}{3}$

Lesson-4: Graphical Equation

After studying this lesson, you should be able to:

- State the nature of graph;
- Solve the equation with the help of graph.

Graphical Equation

In this lesson, we will discuss about graphical equations with the help of following practical examples.

Example–1:

Solve the equation graphically, $2x+5y = 12$ and $y–x = 1$

Solution:

$2x+5y = 12$ ....(i)

$y – x = 1$ ....(ii)

From equation (i), $5y = 12 – 2x.$

$$\text{So, } y = \frac{12 - 2x}{5}$$

So for x = 0, 1, 2 and 3; y = 2.4, 2, 1.6 and 1.2 respectively.

Plotting the points (0, 2, 4), (1, 2), (2, 1.6), (2, 1.6), (3, 1.2) and jointing them, we obtain graph of $2x + 5y = 12$, which is AB in the following figure - 5.1.

Again, from equation (ii), $y – x = 1$
So, $y = 1+x$

So, for x = 0, 1, 2 and 3; y = 1, 2, 3 and 4. Plotting the points (0, 1), (1,2), (3, 4) and jointing them, we obtain graph of $y – x = 1$, which is CD in the following figure-5.1

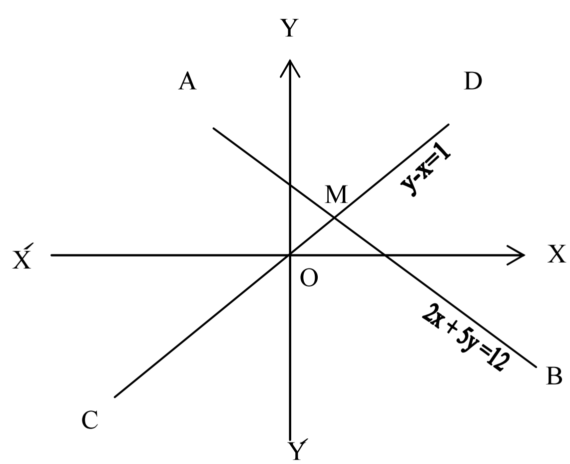

Figure-5.1

From the above graph it is clear that *AB* is the line of equation (i) and *CD* is the line of equation (ii) they are intersecting each other at the point *M*. The co-ordinates of *M* are (1, 2). Hence the required equation is $x = 1$ and $y = 2$.

This gives the solution of $x = 1$, $y = 2$ for the pair of simultaneous equations $2x + 5y = 12$ and $y - x = 1$.

Example – 2:

Draw the graph and solve the following equations; $4y - y + 11$ and $24x - 6y + 2 = 0$

Solution:

The given equations are $4x - y + 11 = 0$

or, $-y = -11 - 4x$

or, $-y = -(11 + 4x)$

so, $y = 11 + 4x$ .... (i)

and $24x - 6y + 2 = 0$

or, $-6y = -24x - 2$

or, $-6y = -(24x+2)$

or, y $\frac{24x+2}{6}$ ....(ii)

We put the values 0, 1, 2 and 3 to *x,* and find corresponding values of *y* then putting down the values in the following table:

x	0	1	2	3
y = 11 + 4x	11	15	19	23
y = (24x + 2)/6	0.33	4.67	8.67	12.67

The system of equations has no solution, because the two lines of the equations are parallel and distinct.

The system of equations has no solution, because the two lines of the equations are parallel and distinct. Hence the given equations are inconsistent.

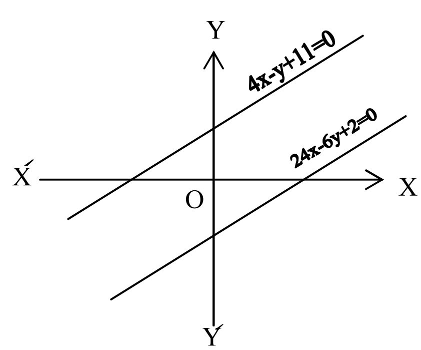

Figure-5.2

The system of equations has no solution, because the two lines of the equations are parallel and distinct. Hence the given equations are inconsistent.

Example–3:

Solve the following quadratic equation graphically;

$x^2 - 4x + 3 = 0$

Solution:

Let $y = x^2 - 4x + 3$

We give values 0, 1, 2, 3, 4 .... to x and find corresponding values of y put down in the following table:

x	0	1	2	3	4
x^2	0	1	4	9	16
$-4x$	0	–4	–8	–12	–16
+3	3	3	3	+3	3
y	3	0	–1	0	3

Plotting the points (0, 3), (1, 0), (2, –1), (3, 0), (4, 3) and jointing them, we get the graph $y = x^2 - 4x + 3$ on shown in the following figure 5.3.

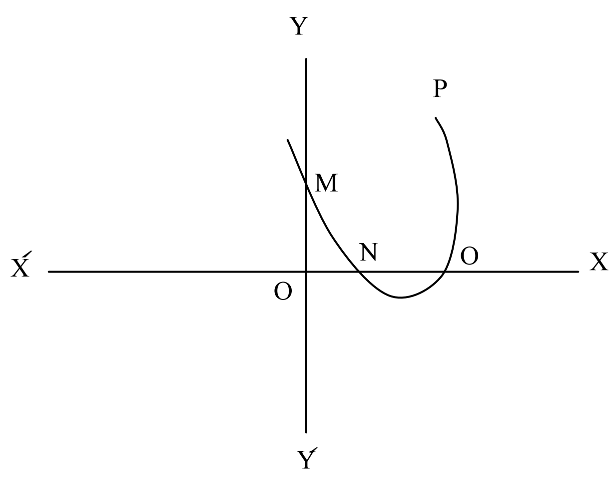

Figure-5.3

The curve MNOP is the graph of $x^2 - 4x + 3 = 0$. The graph of the function $y = x^2 - 4x + 3$ is the curve MNOP which crosses x axis at the points N and O where $x = 1$ and $x = 3$. Hence $x = 1$ and $x = 3$ are the solutions of the quadratic equation $x^2 - 4x + 3 = 0$.

Question for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Draw the graphs and solve the following questions:

   (a) $12x - 7y = -6$
   $5x - 7y = 27$

   (b) $5x + 2y = 6$
   $y = 9x^2 + 16$

2. Draw the graph of the solution set for the system of liner inequalities $x \geq 0$

   $y \geq 0$

   $x + 2y \leq 45$

   $2x + y \leq 60$

3. Draw the graph of the solution set for the system of liner inequalities $2x + y \leq 30$

   $x - 2y \geq 20$

   $-4x + y \leq 39$

4. Draw the graphs and solve the equations

   (i) $x^2 - 5x + 6 = 0$
   $x + 2y + 7 = 0$

   (ii) $3x - 2y + 5 = 0$

Lesson–5: Quadratic Equation

After studying this lesson, you should be able to:

- State the nature of quadratic equation;
- Explain the relationship between roots and coefficient of quadratic equation;
- Explain the formation of quadratic equation; and
- Solve the quadratic equation.

Nature of Quadratic Equation

Generally an equation contains the square of unknown variable is called a quadratic equation.

Generally an equation contains the square of unknown variable is called a quadratic equation. The general method of solving a quadratic equation of the form $ax^2 + bx + c = 0$ is given.

Since $ax^2 + bx + c = 0$; by transposition $ax^2 + bx = c$. Hence dividing both sides by a, the co-efficient of x^2 we have,

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Adding to both sides $\left(\frac{b}{2a}\right)^2$

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\text{or, } x^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = -\frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{or, } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{or, } x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus the required roots of $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Relationship between Roots and the Co-efficient of Quadratic Equation

A quadratic equation has exactly two roots.

A quadratic equation has thus exactly two roots. The relationship between the roots and the co-efficient of the quadratic equations as as follows:

Let the quadratic equation is $ax^2 + bx + c = 0$, then if α and β denote the roots of this quadratic equation, we have

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Therefore by addition,

$$\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{-2b}{2a} = -\frac{b}{a}$$

And by multiplication,

$$\alpha\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$\frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Thus, we have shown that

Sum of the roots $(\alpha+\beta) = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the roots $(\alpha\beta) = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Formation of Quadratic Equation

The formation of the quadratic equation whose roots are given can be explained as follows:

Let the general form of quadratic equation is $ax^2 + bx + c = 0$ and α and β denote the two given roots, where *a, b* and *c* are constants whose values we have to find out.

The sum of the roots $\alpha + \beta = -\frac{b}{a}$,

$$\text{So, } b = -a(\alpha+\beta)$$

And product of the roots, $\alpha\beta = \frac{c}{a}$

$$\text{so, } c = a\,\alpha\beta$$

The above relations imply that

$$ax^2 + bx + c = 0$$

$$\text{or, } ax^2+[-a(\alpha+\beta)]x + a\alpha\beta = 0$$

$$\text{or, } ax^2 - a\alpha x - a\beta x + a\alpha\beta = 0$$

or, $x^2 - x(\alpha+\beta) + \alpha\beta = 0$ [Dividing both sides by 'a']

So, the required quadratic equation will be

x^2 – (sum of the roots)x + (product of the roots) = 0

The following section of this lesson contains some model applications of quadratic equation.

Example–1:

Solve $x^2 - 4x + 13 = 0$

Solution:

$x^2 - 4x + 13 = 0$

We know that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 1, b = –4, c = 13

Substituting the given values we have

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.(1)(13)}}{2.1}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} \quad \left[\text{Since, } i = \sqrt{-1}\right]$$

Therefore, x = 2 + 3i   or, 2 – 3i

Example–2:

Solve $3x + 2\sqrt{x} = \dfrac{10\sqrt{x}}{6x\sqrt{x} - 4x}$

Solution:

We have $3x + 2\sqrt{x} = \dfrac{10\sqrt{x}}{6x\sqrt{x - 4x}}$

$$or,\ 3x + 2\sqrt{x} = \frac{2\sqrt{x}.5}{2x\sqrt{(3x - 2\sqrt{x})}}$$

$$or,\ \left(3x + 2\sqrt{x}\right)\left(3x - 2\sqrt{x}\right) = 5$$

$$or,\ \left[(3x)^2 - 2\left(2\sqrt{x}\right)^2\right] = 5$$

or, $9x^2 - 4x - 5 = 0$

We know that x = $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 9; b = –4; c = –5

Substituting the given values we have

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(9)(-5)}}{2.9}$$

$$= \frac{4 \pm \sqrt{16 + 180}}{18}$$

$$= \frac{4 \pm \sqrt{196}}{18}$$

$$= \frac{4 \pm 14}{18}$$

$$So, x = \frac{18}{18} \quad or, \quad \frac{-10}{18}$$

Example–3:

If α and β are the roots of the equation $x^2 - px + q = 0$,

find the values of (i) $\alpha - \beta$ (ii) $\alpha^2 + \beta^2$ (iii) $\alpha^3 - \beta^3$.

Solution:

Since α and β are the roots of the equation $x^2 - px + q = 0$

So, sum of the roots, $\alpha + \beta = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\dfrac{-P}{1} = P$

Product of the roots, $\alpha\beta = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2} = \dfrac{q}{1} = q$

(i) $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

$= \alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta = (\alpha+\beta)^2 - 4\alpha\beta = p^2 - 4q$

so, $\alpha - \beta = \sqrt{p^2 - 4q}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q$

(iii) $\alpha^3 - \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha + \beta)[(\alpha - \beta)^2 + \alpha\beta]$

$= p(p^2 - 4q + q) = p(p^2 - 3q)$.

Example–4:

The roots of $x^2 - x + 1 = 0$ are α and β; from a quadratic equation whose roots are $\alpha^4 + \beta^4$ and $\alpha^2 + \beta^4$

Solution:

Since α and β are the roots of the equation $x^2 - x + 1 = 0$

Sum of the roots, $\alpha + \beta = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\dfrac{-1}{1} = 1$

And product of the roots, $\alpha\beta = \dfrac{\text{Constant term}}{\text{Coefficient of } x^2} = \dfrac{1}{1} = 1$

Now, the sum of the roots of the required equation,

$= \alpha^2 + \beta^2 + \alpha^2 + \beta^4$

$= (\alpha^2 + \beta^2) + [(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2]$

$= [(\alpha+\beta)^2 - 2\alpha\beta] + [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$= (1^2 - 2.1) + [(1)^2 - 2.1]^2 - (1)^2$

$= -1 + 1 - 2 = -3 + 1 = -2.$

And the product of the roots of the required equation.

$= (\alpha^4 + \beta^2)(\alpha^2 + \beta^4)$

$= \alpha^6 + \alpha^4\beta^4 + \alpha^2\beta^2 + \beta^6$

$= (\alpha^6+\beta^6)(\alpha\beta)^4 + (\alpha\beta)^2$

$= (\alpha^2+\beta^2)(\alpha^4 - \alpha^2\beta^2 + \beta^4) + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ [(\alpha^2+\beta^2)^2 - 3\alpha^2\beta^2] + (\alpha\beta)^2 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ [\{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 3(\alpha\beta)^2] + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(1)^2 - 2(1)]\ [\{(1)^2 - 2(1)\}^2 - 3(1)^2] + (1)^2 + (1)^2$

$= (-1)(1 - 3) + 1 + 1 = 2 + 1 + 1 = 4.$

Hence the required quadratic equation is,

x^2 – (sum of the roots) x + (product of the roots) = 0

or, $x^2 - (-2)x + 4 = 0$

so, $x^2 + 2x + 4 = 0$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is a quadratic equation? Give an example
2. State the characteristics of a quadratic equation.
3. If α and β are the roots of $x^2 - px + q = 0$, form a quadratic equation whose roots are $(\alpha\beta - \alpha - \beta)$.
4. If α and β are the roots of $2x^2 + 3x + 7 = 0$, find the values of (1) $\alpha^2 + \beta^2$ (ii) α/β (iii) β/α (iv) $\alpha^3 + \beta^3$
5. If the equation $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then prove that $a + b + c = 0$ and $a = b = c$.
6. Solve: (i) $3x^2 - 2x - 5 = 0$ (ii) $x^2 + 4x + 4 = y$

Multiple Choice Questions (√ the appropriate answers)

1. The roots of a quadratic equation are 5 and – 2. The equation is:

   (a) $x^2 - 3x - 10 = 0$ (b) $x^2 - 3x + 10 = 0$ (c) $x^2 + 3x - 10 = 0$

2. The quadratic equation whose one root is $3 + 2\sqrt{3}$, is

   (a) $x^2 + 6x - 10 = 0$ (b) $x^2 - 6x - 3 = 0$ (c) $x^2 + 6x + 3 = 0$

3. If α and β are the roots of the question $x^2 - 8x + P = 0$ and $\alpha^2+\beta^2 = 40$ then P is equal to:

   (a) 8 (b) 12 (c) – 12

4. If α and β are the roots of the equation $2x^2 - 4x + 3 = 0$ the value of the $\alpha^3 + \beta^3$ is:

   (a) – 1 (b) 5 (c) 2

5. If α and β are the roots of the equation $2x^2 - 3x + 1 = 0$, then the equation whose roots are α/β and β/α is:

   (a) $2x^2 + 5x + 2 = 0$ (b) $2x^2 - 5x + 2 = 0$ (c) $2x^2 + 5x - 2 = 0$

6. If one root of the equation $3x^2 - 10x + 3 = 0$ is 1/3, then the other root is:

   (a) 3 (b) – 1/3 (c) –3

Lesson-6: Application of Equation in Business Problems

After studying this lesson, you should be able to:

- Apply the principles of equations to solve the business problems

Introduction

The application of equations in the solution of business problems is of great use.

The application of equations in the solution of business problems is of great use. It is usual to take the following steps in the solution of business problems.

(i) Read the problem very carefully and try to understand the relation between the quantities stated therein.

(ii) Denote the unknown quantity by the letter $x, y, z,$ etc.

(iii) Translate the statements of the problem step by step into mathematical statements; to the extent it is possible.

(iv) Solve the equation for the unknown

(v) check whether the obtained solution satisfies the condition given in the problem.

The following section of this lesson contains some model applications of equations to solve business problems.

Example-1:

A man says to his son, "Seven years ago I was seven times as old as you were, and three years hence, I shall be three times as old as you will be". Find their present ages.

Solution:

Let the present age of the son be x years.

His age seven years ago = $x - 7$

The father's age 7 years ago = $7(x - 7)$

Father's present age = $7(x - 7) + 7$.

Son's age 3 years hence = $x + 3$

Father's age 3 years hence = $7(x - 7) + 7 + 3 = 7(x - 7) + 10$

Using the given information we can write

$7(x - 7) + 10 = 3(x + 3)$

or, $7x - 49 + 10 = 3x + 9$

or, $7x - 3x = 49 + 9 - 10$

or, $4x = 48$

so, $x = 12$

So, son's present age = 12 years.

Father's present age = $7(12 - 7) + 7$

$= 35 + 7 = 42$ years.

Example-2:

A man's annual income has increased for Tk.7,500 but there is no change in the income tax payable for him since the rate of income tax has been reduced from 10% to 7%. Find his present income.

Solution:

Let previous annual income = x

$\therefore$ Present income = x + 7500

His income tax of previous year = 10% of x = 0.10x

His income tax for present year = 70% of (x + 7500) = 0.07x + 525

According to the question we can write

$0.10x = 0.07x + 525$

or, $0.10x = 0.07x + 525$

or, $0.03x = 525$

or, $x = (525 \div 0.03)$

so, $x = 17{,}500$

So, the present income = (x + 7500) = (17,500 + 7500) = Tk.25,000.

Example-3:

Tk.20,000 is invested in two shares. The first yield is Tk.12% p.a. and the second yield is 14% p.a. If the total yield at the end of one year is 13% p.a.; how much was invested at each rate?

Solution:

Let x in Tk. be invested at 12%.

Then (Tk.20000 – x) is invested at 14%

Interest at 12% = 12% of x = 0.12x

Interest at 14% = 14% of (20,000 – x) = 2800 – 0.14x

Total Interest on 20,000 at 13% = 13% of 20,000 = 2,600

According to the given information we can write

$260 = 2800 - 0.14x + 0.12x$

or, $0.02x = 2800 - 2600$

or, $x = (200 \div 0.02) = 10{,}000$

Hence Tk.10,000 is invested at 12% and (20,000 – 10,000) = Tk.10,000 is invested at 14%.

Example-4:

Demand and supply equations are $2p^2 + q^2 = 11$ and $p + 2q = 7$ respectively. Find the equilibrium price and quantity. (Where p stands for price and for quantity).

Solution:

The demand function is : $2p^2 + q^2 = 11$ .... (1)

And supply function is : $p + 2q = 7$

So, $p = 7 - 2q$ .... (2)

Putting the value of p in equation (1) we have,

$2\,(7 - 2q)^2 + q^2 = 11$

or, $2\,(49 - 28q + 4q^2) + q^2 = 11$

or, $98 - 56q + 8q^2 + q^2 = 11$

or, $9q^2 - 56q + 98 - 11 = 0$

or, $9q^2 - 56q + 87 = 0$

or, $9q^2 - 27q - 29q + 87 = 0$

or, $9q\,(q - 3) - 29\,(q - 3) = 0$

or, $(q - 3)\,(9q - 29) = 0$

either, $q - 3 = 0$ or, $9q - 29$

so, $q = 3$ or, $q \ \frac{29}{9}$

When $q = 3$; the price is, $p = 7 - 2q = 7 - 2 \times 3 = 7 - 6 = 1$.

When $q = \frac{29}{9}$, the price is, $p = 7 - 2 \times \frac{29}{9} = 7 - \frac{58}{9} = \frac{63 - 58}{9} = \frac{5}{9}$

Therefore the equilibrium price and quantity are:

$(p, q) = (1, 3)$ or $\left(\frac{5}{9}, \frac{29}{9}\right)$

Questions for Review:

These questions are designed to help you assess how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. Mr. Sajib invested Tk.16,000 in two types of debentures of Tk.100 each. 13% debentures were purchased at Tk.150 each and 10% debentures were purchased at Tk.120 each. If he got interest of Tk.970 after 1 year, find the sum invested in each type of debenture.
2. The ages (in years) of Ramesh and Rahim are in the ratio 5 : 7. If Ramesh were 9 years older and Rahim were 9 years younger, the age of Ramesh would have been were twice the age of Rahim. Find their ages.
3. If the total manufacturing cost 'y' of making x units of a toy is: $y = \frac{25x - 9000}{2}$
   (i) What is the variable cost per unit?
   (ii) What is the fixed cost?
   (iii) What is the average cost of manufacturing 5,000 units?
   (iv) What is the marginal cost of producing 3,000 units?
   (v) What will be the effect on average cost per unit if volume changes?

Multiple choice questions (√ the appropriate answers)

1. The total cost of 6 books and 4 pencils is Tk.34 and that of 5 books and 5 pencils is Tk.30. The costs of each book and each pencil respectively are:

   a) Tk.5 and Tk.1 b) Tk.1 and Tk.5 c) Tk.6 and Tk.1

2. If 3 chairs and 2 tubes cost Tk.1200 and 5 chairs and 3 tubes cost Tk.1900, then the cost of 2 chairs and 2 tubes is:

   a) Tk.1000 b) Tk.900 c) Tk.700

3. The monthly income of A & B are in the ratio 4 : 3, Each of them saves Tk.600. If the ratio of the expenditures is 3 : 2, then the monthly income of A is:

   a) Tk.1800 b) Tk.2400 c) Tk.2000

4. If the laws of demand and supply are respectively given by the equation $4q + 9p = 48$ and $p = \frac{q}{9} + 2$, then the value of the equilibrium price and quantity respectively are:

   a) $^8/_3$ and 6 b) $^3/_8$ and 5 c) 8 and 6