Matrix Algebra

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This unit is designed to introduce the learners to the basic concepts associated with matrix algebra. The learners will learn about different types of matrices, operations of matrices, determinant and matrix inversion. This unit also discusses the procedure of determining the solution of the system of linear equations by using inverse matrix method, Gaussian method and Cramer's' Rule. Some relevant business and economic applications of matrix algebra are also provided in this unit for clear and better understanding of the learners.

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Lesson-1: Matrix: An Introduction

After studying this lesson, you should be able to:

- State the nature of a matrix;
- Explain matrix representation of data.
- Define different types of matrices.

Introduction

J. J. Sylvester was the first to use the word 'matrix' in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. Matrix is a powerful tool of modern mathematics and its study is becoming important day by day due to its wide applications in every branch of knowledge. Matrix arithmetic is basic to many of the tools of managerial decision analysis. It has an important role in modern techniques for quantitative analysis of business and economic decisions. The tool has also become quite significant in the functional business and economic areas of accounting, production, finance and marketing.

Matrix arithmetic is basic to many of the tools of managerial decision analysis.

Matrix

Whenever one is dealing with data, there should be concern for organizing them in such a way that they are meaningful and can be readily identified. Summarizing data in a tabular form can serve this function. A matrix is a common device for summarizing and displaying numbers or data. Thus, a matrix is a rectangular array of elements and has no numerical value. The elements may be numbers, parameters or variables. The elements in horizontal lines are called rows, and the elements in vertical lines are called columns.

A matrix is a rectangular array of elements and has no numerical value.

A matrix is characterized further by its dimension. The dimension or order indicates the number of rows and the number of columns contained within the matrix. If a matrix has *m* rows and *n* columns, it is said to have dimension ($m \times n$), which is read as: *m* by *n*.

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Types of Matrices:

Row Matrix: The matrix with only one row is called a row matrix or row vector.

For example: $A = \begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$.

Column Matrix: The matrix with only one column is called a column matrix or column vector.

For example: $A = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Row matrix and column matrix are usually called as row vector and column vector respectively.

Square Matrix: If the number of rows and the number of columns of a matrix are equal then the matrix is of order $n \times n$ and is called a square matrix of order *n*.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Rectangular Matrix: If the number of rows and the number of columns of a matrix are not equal then the matrix is called a rectangular matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Singular matrix: A square matrix A is said to be singular if the determinant formed by its elements equal to zero.

For example: Let $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$.

Determinant of $A = |A| = (2 \times 2) - (4 \times 1) = 0$.

Hence A is a singular matrix.

Non-singular Matrix:

A square matrix A is said to be non-singular if the determinant formed by its elements is non-zero.

For example: $A = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$

$|A| = (5 \times 4) - (3 \times 2) = 20 - 6 = 14$.

Hence A is a non-singular matrix.

Null or Zero Matrix: The matrix with all of its elements equal to zero is called a null matrix or zero matrix.

For example: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Diagonal Matrix: A matrix whose all elements are zero except those in the principal diagonal is called a diagonal matrix.

For example: $A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example: $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Sub-Matrix: A matrix that is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub-matrix of the given matrix.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a sub-matrix of $B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{pmatrix}$

Unit matrix or Identity matrix: A matrix with every element in the principal diagonal equals to one and all other elements equal to zero is called a unit matrix. A unit matrix is a square matrix. It is denoted by *I*.

For example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equal Matrix: Two matrices *A* and *B* are said to be equal if their corresponding elements are equal.

For example: Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ then $A = B$

Transpose of a Matrix: If the columns of a given matrix *A* are changed into rows or the rows are changed into columns, the matrix thus formed is called the transpose of the matrix *A* and it is generally denoted by A^T.

For example: Let $\mathrm{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

Symmetric Matrix: A square matrix *A* is called symmetric if it be same as its transpose so that $A = A^T$.

For Example: Let $\mathrm{A} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ then $\mathrm{A}^{\mathrm{T}} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

i.e., $A = A^T$, so *A* is a symmetric matrix.

Skew-Symmetric Matrix: A square matrix *A* is called skew-symmetric if $A^T = -A$.

For example: Let $\mathrm{A} = \begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{pmatrix}$

then $A^T = \mathrm{A}^{\mathrm{T}} = \begin{pmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{pmatrix} = -A$

i.e., $A^T = -A$, hence *A* is a skew-symmetric matrix.

Involuntary Matrix: A square matrix *A* is called involuntary matrix provided it satisfies the relation $A^2 = I$, where *I* is the identity matrix.

For example: $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Idempotent Matrix: A square matrix A is called idempotent matrix provided it satisfies the relation $A^2 = A$.

Example: $A = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

Nilpotent Matrix: A square matrix A is called nilpotent matrix of order ***m*** provided it satisfies the relation $A^m = 0$ and $A^{m-1} \neq 0$, where m is a positive integer and 0 is the null matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{pmatrix}$ since $A \neq 0, A^2 = 0$

Complex Conjugate of a Matrix: It is a matrix obtained by replacing all its elements by their respective complex conjugates.

For example: If $A = \begin{pmatrix} 2 & +3i & 5 \\ 3 & -3i & 7 \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 2 & -3i & 5 \\ 3 & +3i & 7 \end{pmatrix}$

Hermitian Matrix: A matrix having complex elements of a square matrix A is a Hermitian matrix. If $(A)' = A$, then A is called Hermitian matrix.

Skew-Hermitian Matrix: A matrix having complex elements for matrix $A.(A) = -A.A$ is skew hermitian matrix.

Co-factor Matrix

A matrix, which is formed by the co-factors of the corresponding elements, is called co-factor matrix and is denoted by A^C.

For example: If a matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then, the co-factor matrix, $A^C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

Adjoint Matrix:

The Adjoint matrix is the transpose of the co-factor matrix, that is $adjA = A_j = (cof\ A)^T$

Orthogonal Matrix: A square matrix A is called an orthogonal matrix if $AA^T = A^T A = I$, where I is an identity matrix and A^T is the transpose matrix of A.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you understand by matrix?
2. Why matrix algebra is so important in business and economics? Explain.
3. Discuss the various types of matrices.
4. In an examination, 20 students from college A, 30 students from college B and 40 students from college C appeared. Only 15 students from each college could get through the examination. Out of them 10 students from college A, 5 students from college B and 10 students from college C secured full marks. Write down the above data in matrix form.

Lesson-2: Matrix Operations

After studying this lesson, you should be able to:

- Express the concept of matrix operations;
- Add or subtract given matrices;
- Multiply given matrices.

Introduction

The operations of matrices are addition, subtraction, multiplication and division of which addition and multiplication are the main operations. In this lesson we will discuss some of the operations of matrix algebra.

Matrix Addition

Two matrices of the same dimensions are said to be conformable for addition.

Two matrices of the same dimensions are said to be conformable for addition. The addition is performed by adding corresponding elements from the two matrices and entering the result in the same row-column position of a new matrix.

If A and B are two matrices, each of size $m \times n$ then the sum of A and B is the $m \times n$ matrix C whose elements are $C_{ij} = A_{ij} + B_{ij}$; i = 1, 2, 3… m and j = 1, 2, 3, ….n.

Properties of Matrix Addition:

- *Commutative law*: Matrix addition is commutative. If A and B are two matrices of same order $m \times n$, then $A + B = B + A$.
- *Associative law*: Matrix addition is associative. If A, B and C are three matrices of same order $m \times n$, then $A + (B + C) = (A + B) + C$.
- *Distributive law*: If A and B are two matrices of same order $m \times n$, and K is any scalar, then $K(A + B) = KA + KB$.
- *Existence of additive identity*: If O denotes null matrix of the same order as that of A, then $A + O = A = O + A$.
- *Existence of an additive inverse*: If A be any given $m \times n$ matrix and there exists another $m \times n$ matrix B such that $A + B = O = B + A$; where O be the $m \times n$ null matrix.
- *Cancellation law*: If A, B and C are three matrices of same order $(m \times n)$, then $A + C = B + C \Rightarrow A = B$.

Example-1:

Find the sums $A + B$ of the following matrices

$$A = \begin{pmatrix} 8 & 9 \\ 12 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 13 & 4 \\ 2 & 6 \end{pmatrix}$$

Solution:

$$A + B = \begin{pmatrix} 8+13 & 9+4 \\ 12+2 & 7+6 \end{pmatrix} = \begin{pmatrix} 21 & 13 \\ 14 & 13 \end{pmatrix}$$

Matrix Subtraction

The subtraction of two matrices is possible only when they are of the same order. Such matrices are said to be conformable for subtraction. The subtraction is performed by subtracting corresponding elements of the two matrices and entering the result in the same row-column position of a new matrix.

The subtraction of two matrices is possible only when they are of the same order.

If A and B are two matrices, each of size $m \times n$ then the subtraction of A and B is the $m \times n$ matrix C whose elements are $C_{ij} = A_{ij} - B_{ij}$; $i = 1, 2, 3\ldots\ m$ and $j = 1, 2, 3, \ldots.n$.

Example-2:

Find the difference $A - B$ of the following matrices

$$A = \begin{pmatrix} 3 & 7 & 11 \\ 12 & 9 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & 8 & 1 \\ 9 & 5 & 8 \end{pmatrix}$$

Solution:

$$A - B = \begin{pmatrix} 3-6 & 7-8 & 11-1 \\ 12-9 & 9-5 & 2-8 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 10 \\ 3 & 4 & -6 \end{pmatrix}$$

Scalar Multiplication of a Matrix

A matrix can be multiplied by a constant by multiplying each component in the matrix by the constant. The result is a new matrix of the same dimensions as the original matrix.

A matrix can be multiplied by a constant by multiplying each component in the matrix by the constant.

If K is any real number and $A = [a_{ij}]$ is an $m \times n$ matrix, then the product KA is defined to be the matrix whose components are given by K times the corresponding component of A, i.e., $KA = [Ka_{ij}]$

Laws of scalar multiplication:

(i) $K(A + B) = KA + KB$

(ii) $(K_1 + K_2)A = K_1 A + K_2 A$

(iii) $IA = A$

(iv) $(K_1 K_2)A = K_1(K_2 A)$.

Example-3:

If $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, Find $5A$.

Solution:

$$5A = 5\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 5 \\ 10 & 5 & 10 \\ 15 & 10 & 5 \end{pmatrix}$$

Multiplication of Matrices

If the number of columns of the first matrix is equal to the number of rows of the second matrix, such matrices are said to be conformable for multiplication.

If the number of columns of the first matrix is equal to the number of rows of the second matrix, such matrices are said to be conformable for multiplication. Let A be a matrix of order $m \times p$ and B be a matrix of order $p \times n$. Then the product AB is defined to be a matrix C of order $m \times n$.

Properties of Matrix Multiplication

- *Associative law*: Multiplication of matrices is associative i.e. $A(BC) = (AB)C$.
- *Distributive law*: Multiplication of matrices is distributive with respect to matrix addition i.e. $A(B + C) = AB + AC$.
- *Multiplication of a matrix by a null matrix:* If A is $n \times m$ and O is $m \times n$ matrices, then $AO = O = OA$.
- *Multiplication of a matrix by a unit matrix:* If A is a square matrix of order $n \times n$ and I is the unit matrix of same order, then $IA = A = AI$.
- *Multiplication of matrix by itself:* If A is a square matrix then $A.A = A^2$.

Example-4:

Find AB, where $A = [9 \quad 11 \quad 3]$ and $B = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

Solution:

The matrices A and B are conformable for multiplication. The dimensions of A and B are 1×3 and 3×1 respectively, i.e., the product matrix AB will be 1×1 and a scalar, derived by multiplying each element of the row vector by its corresponding element in the column vector and then summing the products.

$$AB = [(9 \times 2) + (11 \times 6) + (3 \times 7)] = 18 + 66 + 21 = 105.$$

Example-5:

If $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

Find (i) $3A - 4B$

(ii) $2A - 3B$

Solution:

(i) $3A - 4B = 3\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{pmatrix} - \begin{pmatrix} 4 & 8 & -4 \\ 0 & -4 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 6-4 & 9-8 & 3-(-4) \\ 0-0 & -3-(-4) & 15-12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

(ii) $2A - 3B = 2\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 6 & 2 \\ 0 & -2 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -3 \\ 0 & -3 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 6-6 & 2-(-3) \\ 0-0 & -2-(-3) & 10-9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

Example-6:

If $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$

then find *AB*. Whether *BA* exists? Give reason.

Solution:

$$AB = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3.1+1.2+2.1 & 3.4+1.2+2.0 \\ 0.1+1.2+1.1 & 0.4+1.2+1.0 \\ 1.1+2.2+0.1 & 1.4+2.2+0.0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{pmatrix}$$

Here A is a matrix of order 3×3 and B is a matrix of order 3×2. Hence BA does not exist as number of columns in B is not equal to the number of rows in A.

Example-7:

If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$.

Find AB and show that $AB \neq BA$

Solution:

$$AB = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.2 + (-2).4 + 3.2 & 1.3 + (-2).5 + 3.1 \\ -4.2 + 2.4 + 5.2 & -4.3 + 2.5 + 5.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 \\ 10 & 3 \end{pmatrix}$$

and $BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 2.1 + 3.(-4) & 2.(-2) + 3.2 & 2.3 + 3.5 \\ 4.1 + 5.(-4) & 4.(-2) + 5.2 & 4.3 + 5.5 \\ 2.1 + 1.(-4) & 2.(-2) + 1.2 & 2.3 + 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{pmatrix}$$

Hence, $AB \neq BA$.

Example 8:

Evaluate $A^2 - 4A - 5I$, where $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

$$A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 - 4 + 5 & 8 - 8 + 0 & 8 - 8 + 0 \\ 8 - 8 + 0 & 9 - 4 - 5 & 8 - 8 + 0 \\ 8 - 8 + 0 & 8 - 8 + 0 & 9 - 4 - 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \phi, \text{ where } \phi \text{ is a null matrix.}$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. If $A = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, find $A^2 + 3A + 5I$ where I is unit matrix of order 2.

2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Find a matrix C such that $A + B = 2C$.

3. If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, find A^3.

4. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 & 9 \\ 6 & -2 & 1 \end{pmatrix}$

   (i) Write down the order of the matrices A and B.
   (ii) Write down the order of the product AB.
   (iii) Calculate AB.
   (iv) Is it possible to calculate BA?
   (v) *Is* $AB = BA$?
   (vi) Are the following possible for operation?
   $A + B$, $A - B$, $2B$ and A^2

Lesson-3: Determinant

After studying this lesson, you should be able to:

- State the concept of determinant;
- Describe the advantages of determinant;
- Express the Cramer's rule;
- Solve the system of linear equations by Cramer's Rule.

Introduction

The present lesson is devoted to a brief discussion of determinants and their more elementary properties. The determinant concept is of a particular interest in solving simultaneous equations.

Determinant

An important concept in matrix algebra is that of the determinant. If a matrix is square, the elements of the matrix may be combined to compute a real-valued number called the determinant and is denoted either by the symbol Δ, or by placing vertical lines around the elements of the matrix (like **|A|**) or simply by *det.A*. The signs of the successive terms in the expansion of determinants will be alternately positive and negative until the last term is reached.

If a matrix is square, the elements of the matrix may be combined to compute a real-valued number called the determinant.

$$\text{If, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinant of A will be denoted by $\Delta = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Types of Determinants

First Order Determinant: A determinant of the first order is defined by the determinant of a 1×1 Matrix. The determinant of a 1×1 matrix is simply the value of the one element contained in the matrix.

Let, $A = [a_{11}]$ be a square matrix. Then $|A| = a_{11}$ be a determinant of first order.

Second Order Determinant: A determinant of the second order is defined by the determinant of a 2×2 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

That is the value of the determinant is given by the difference of the cross products.

Third Order Determinant: A determinant of the third order is defined by the determinant of a 3×3 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minors and Co-factors: The method discussed earlier applies for calculating the determinant of a 2×2 or 3×3 matrix. It does not, however, apply to matrices of higher dimensions. It is required a procedure for calculating a determinant that applies to any square matrix. This procedure is termed as the method of co-factor expansion. Before discussing the method of co-factor expansion, we must define two terms minor and co-factor.

Minors

The minor of an element is defined as a determinant by omitting the row and the column containing the element.

The minor of an element is defined as a determinant by omitting the row and the column containing the element. Thus, a minor is the determinant of the sub matrix formed by deleting the *i*-th row and *j*-th column of the matrix.

If a matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then – minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

Co-factors

A co-factor is a minor with a prescribed sign.

The co-factor of an element is the co-efficient of the element in the expanded form and is equal to the corresponding minor with proper sign. Thus, a co-factor is a minor with a prescribed sign. The rules for the sign of a co-factor of any element = $(-1)^{i+j} \times$ its minor, where i = number of row and j = number of column.

The co-factor of $a_{ij} = c_{ij} = (-1)^{i+j} M_{ij}$

For example, co-factor of $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

co-factor of $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

Example-1:
Find the minors and co-factors of the elements at the 1st row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 7 \end{vmatrix}$$

Solution:

The minor of the element 1, i.e., a_{11} is $M_{11} = \begin{vmatrix} 5 & 0 \\ 2 & 7 \end{vmatrix} = 35$

The minor of the element 2, i.e., a_{12} is $M_{12} = \begin{vmatrix} 4 & 0 \\ 3 & 7 \end{vmatrix} = 28$

The minor of the element 3, i.e., a_{13} is $M_{13} = \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = -7$

The co-factor of 1, i.e., a_{11} is $C_{11} = (-1)^{1+1} .35 = 35$

The co-factor of 2, i.e., a_{12} is $C_{12} = (-1)^{1+2} .28 = 28$

The co-factor of 3, i.e., a_{13} is $C_{13} = (-1)^{1+3} (-7) = -7$

Expansion of Determinant and Use of Sarrus Diagram

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

If the co-factor of a_{11}, a_{12} and a_{13} are A_{11}, A_{12} and A_{13} respectively, then

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

Sarrus Diagram: We can find out determinant value of a given matrix very conveniently by using Sarrus diagram. It is found by the following process:

We can find out determinant value of a given matrix by using Sarrus diagram.

(i) Rewrite the first two columns of the matrix to the right of the original matrix.

(ii) Locate the elements on the three primary diagonals (P_1, P_2, P_3) and those on the three secondary diagonals (S_1, S_2, S_3).

(iii) Multiply the elements on each primary and each secondary diagonal.

(iv) The determinant equals the sum of the products for the three primary diagonals minus the sum of the products for the three secondary diagonals.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant may be found by the following process

$$\begin{array}{ccccc} & & & S_1 \quad S_2 & S_3 \\ \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right\| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \right| \\ & & & P_1 \quad P_3 & P_2 \end{array}$$

Thus, algebraically the determinant value is computed as

$$|A| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Hence expansion of the determinant of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ will be

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$
$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Example-2:

Find the value of $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

Solution:

Let D = $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

$= 1\,(0 - 5) - 5\,(-4 + 20) + 3\,(2 - 0)$

$= (-5 - 80 + 6) = 79.$

Properties of Determinants

Certain properties hold for determinants. The following properties can be useful in computing the value of the determinant.

- If two rows or columns are interchanged in a determinant, the sign of the determinant changes but its value is unchanged.
- If rows are changed into columns and columns into rows, the determinant remains unchanged.
- If two rows or columns are identical in a determinant, it vanishes.

- If all the elements of any row or column are zero, the determinant is zero.
- If any multiple of one row or column is added to another row or column, the value of the determinant is unchanged.
- If any row or column is a multiple of another row or column, the determinant equals to zero.

Example-3:

Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

Applying $C'_1 = C_1 - C_2$; $C'_2 = C_2 - C_3$ we get

$$\begin{vmatrix} 0 & 0 & 1 \\ a - b & b - c & c \\ a^2 - b^2 & b^2 - c^2 & c^2 \end{vmatrix}$$

$$= (a-b)\,(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$= (a-b)\,(b-c)\,(c-a)$$

Example-4:

Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution:

Applying $C'_1 = C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R'_1 = R_1 - R_2$; $R'_2 = R_2 - R_{23}$

$$= 2(a+b+c)\begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & (a+b+c) & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)^3$$

Example-5:

Show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

Solution:

Applying $C_3' = C_2 + C_3$ we get,

$$= \begin{vmatrix} 1 & x & x+y+z \\ 1 & y & x+y+z \\ 1 & z & x+y+z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$= 0.$

Example-6:

Solve the equation $\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ x^3 & a^3 & b^3 \end{vmatrix} = 0$

Solution:

Applying $C'_2 = C_2 - C_1$; $C'_3 = C_3 - C_2$; we get

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a-x & b-a \\ x^3 & a^3-x^3 & b^3-a^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a-x & b-a \\ a^3-x^3 & b^3-a^3 \end{vmatrix} = 0$$

or, $(a-x)(b-a)(b^2+ab+a^2-a^2-ax-x^2) = 0$

or, $(a-x)(b-a)(b^2+ab-ax-x^2) = 0$

or, $-(a-x)(b-a)(x^2+ax-ab-b^2) = 0$

or, $(a-x)(b-a)(x^2+ax-ab-b^2) = 0$

$\therefore\ x = a \ \ or \ \ x = \dfrac{-a \pm \sqrt{a^2 - 4(-ab-b^2)}}{2}$

or, $x = a \ \ or \ \ x = \dfrac{-a \pm \sqrt{a^2 + 4ab + b^2)}}{2}$

$\therefore\ x = a, b, -(a+b)$

Cramer's Rule and Its use in the Solution of Equations

Cramer's rule is a simple rule using determinants to express the solution of a system of linear equations for which the number of equations is equal to the number of variables. This rule states $\bar{x}_i = \frac{D_i}{D}$ where x_i is the *i*-th unknown variable in a series of equations, D is the determinant of the coefficient matrix, and D_i is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector of constants. Thus, Cramer's rule can be fruitfully applied in case $D \neq 0$.

Example-7:

Solve the following system of equations by using Cramer's Rule.

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

Solution:

Here $D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 419$

$$D_x = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257$$

$$D_y = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676$$

$$D_z = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$$

We know from the Cramer's Rule, $\frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D}$

Hence $x = \frac{D_x}{D} = \frac{1257}{419} = 3$

$$y = \frac{D_y}{D} = \frac{1676}{419} = 4$$

$$z = \frac{D_z}{D} = \frac{2514}{419} = 6.$$

Example-8:

Solve the following system of equations by using Cramer's Rule.

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

Solution:

Here $D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = -4$

$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = -4$

$D_y = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = -12$

$D_z = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = -20$

We know from the Cramer's Rule, $\dfrac{x}{D_x} = \dfrac{y}{D_y} = \dfrac{z}{D_z} = \dfrac{1}{D}$

Hence $x = \dfrac{D_x}{D} = \dfrac{-4}{-4} = 1$

$y = \dfrac{D_y}{D} = \dfrac{-12}{-4} = 3$

$z = \dfrac{D_z}{D} = \dfrac{-20}{-4} = 5.$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find all the minors and co-factors of the following determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -2 & 8 & 1 \end{vmatrix}$$

2. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

4. Find the value of $\begin{vmatrix} x+y & x & y \\ x & x+z & z \\ y & z & y+z \end{vmatrix}$

5. Solve the following system of equations by using Cramer's Rule:

$$x+5y-z=9$$
$$3x-3y+2z=7$$
$$2x-4y+3z=1$$

6. Solve the following system of equations by using Cramer's Rule:

$$x-y+z=1$$
$$x+y-2z=0$$
$$2x-y-z=0$$

7. Solve the equation $\begin{vmatrix} p+x & q+x & r+x \\ q+x & r+x & p+x \\ r+x & p+x & q+x \end{vmatrix} = 0$

Lesson-4: Matrix Inversion

After studying this lesson, you should be able to:

- Explain inverse matrix;
- Solve system of linear equations by inverse matrix method.

Introduction

The operation of dividing one matrix directly by another does not exist in matrix theory but equivalent of division of a unit matrix by any square matrix can be accomplished (in most cases) by a process known as inversion of matrix. The concept of inverse matrix is useful in solving simultaneous equations, input-output analysis and regression analysis.

If A is a square matrix of order n, then a square matrix B of the same order n is said to be inverse of A if AB = BA = I (unit matrix).

Inverse Matrix

If A is a square matrix of order n, then a square matrix B of the same order n is said to be inverse of A if $AB = BA = I$ (unit matrix).

Methods of Matrix Inversion

There are several methods for determining the inverse of a matrix; two of these are discussed in below.

(i) Co-factor matrix method.
(ii) Gauss- Jordan Elimination method.

Working Rule for Inverse Matrix (Co-factor matrix method)

To evaluate the inverse of a square matrix A, we should follow the following steps:

(i) Evaluate |A| for the matrix A and be sure that $|A| \neq 0$

(ii) Calculate the co-factors of all the elements of the matrix A.

(iii) Find the matrix of the co-factor A^C.

(iv) Then find the Adjoint of A by taking transpose of A^C such that Adj A = $(A^C)^T$.

(v) Finally divide all the elements of Adj A by |A| to get the required inverse A^{-1}.

Example-1:

Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} = 4 \neq 0$

The co-factor matrix of A is, $A^C = \begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix}$

The Ad joint matrix of A is, $A^j = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^{j} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$$

Example-2:

Find the inverse of the matrix, A = $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 1$

The co-factor matrix of A is, $A^{C} = \begin{bmatrix} 3 & -1 & 3 \\ -4 & 2 & -5 \\ -2 & 1 & -2 \end{bmatrix}$

The Adjoint matrix of A is, $A^{J} = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^{J} = \frac{1}{1}\begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$$

Gauss-Jordan Elimination Method

To determine the inverse of an $m \times m$ matrix 'A', following are the steps

(i) Determining the determinant value of A, whether it is non-singular or not.

(ii) Augmenting the matrix A with an $m \times m$ identity matrix, resulting in ($A \mid I$).

(iii) Performing row operations on the entire augmented matrix so as to transform 'A' into an $m \times m$ identify matrix. The resulting matrix will have the following form ($I \mid A^{-1}$) where, the A^{-1} can be read to the right of the vertical line.

Example-3:

Find the inverse of the matrix, $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Solution:

Augmented the matrix 'A' by 2×2 identity matrix, we get –

$$\left[\begin{array}{cc|cc} 3 & 7 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{3} & \frac{1}{3} & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \text{applying } r_1^{/} = r_1 \times \frac{1}{3}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{2}{3} & 1 \end{array}\right] \text{applying, } r_2^{/} = r_2 - r_1 \times 2$$

$$\left[\begin{array}{cc|cc} 1 & \frac{7}{3} & \frac{1}{3} & 0 \\ 0 & 1 & -2 & 3 \end{array}\right] \text{applying, } r_2^{/} = r_2 \times \frac{1}{3}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & -7 \\ 0 & 1 & -2 & 3 \end{array}\right] \text{applying, } r_1^{/} = r_1 - r_2 \times \frac{7}{3}$$

So, the inverse of 'A' is, $A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Solution of Linear Equations by Using Inverse of Matrix

Matrix algebra permits the concise expression of a system of linear equations.

Matrix algebra permits the concise expression of a system of linear equations. The inverse matrix can be used to solve a system of simultaneous equations. Let a system of linear equations are:

$$a_{11}x + a_{12}y + a_{13}z = k_1$$
$$a_{21}x + a_{22}y + a_{23}z = k_2$$
$$a_{31}x + a_{32}y + a_{33}z = k_3$$

It can be written in the matrix form as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$AX = B$; where, $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$$X = A^{-1}B$$

Example-4:

Use matrix inversion to solve the following system of equations

$$4x_1 + x_2 - 5x_3 = 8$$
$$-2x_1 + 3x_2 + x_3 = 12$$
$$3x_1 - x_2 + 4x_3 = 5$$

Solution:

The given system of equations can be written in the matrix form

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{vmatrix} = 98$$

The co-factor matrix of A is $A^C = \begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$

The Adjoint matrix of A is, $A_j = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$\therefore$ The inverse of A is, $A^{-1} = \frac{1}{\Delta}A_j = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$$X = A^{-1}B = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$\therefore$ $x_1 = 2, x_2 = 5, x_3 = 1.$

Example-5:

Solve the following system of equations by using Gaussian method.

$$x + y + z = 7$$
$$x + 2y + 3z = 16$$
$$x + 3y + 4z = 22$$

Solution:

Given system of equations in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix}$$; Applying $R_2 = R_2 - R_1$; $R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -3 \end{bmatrix}$$; Applying $R_3 = R_3 - 2R_2$

$$x + y + z = 7$$

Hence $$y + 2z = 9$$

$$-z = -3$$

Thus, $z = 3$, $y = 3$, $x = 1$.

Example-6:

Solve the following system of equations by using Gaussian method.

$$2x - 5y + 7z = 6$$
$$x - 3y + 4z = 3$$
$$3x - 8y + 11z = 11$$

Solution:

Given system of equations in matrix form

$$\begin{bmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$; Applying $R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$; Applying $R_3 = R_3 - R_2$

$$x - 3y + 4z = 3$$

Hence $$y - z = 0$$

$$0 = 2$$

Since, 0 = 2 is false, the given system of equations has no solution. So given system of equations is inconsistent.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 2 \end{bmatrix}$

2. Find the inverse of the matrix, $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

3. Solve the following system of equations by using Gaussian method.

$$2x - 5y + 7z = 6$$
$$x - 3y + 4z = 3$$
$$3x - 8y + 11z = 11$$

4. Use matrix inversion to solve the following system of equations:

$$\mathrm{x} + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

5. Use matrix inversion to solve the following system of equations:

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Lesson-5: Application of Matrices in Business

After studying this lesson, you should be able to:

- Develop matrices by using given business information;
- Apply the concepts of matrices to solve the business problems.

Introduction

Matrix algebra permits the concise expression of a system of linear equations.

Matrix is the powerful tool in modern mathematics having wide applications. Demographers, sociologists, economists use matrices in different way. Many economic relationships can be approximated by linear equations. Matrix algebra permits the concise expression of a system of linear equations. Let us know few applications of matrices in business.

Illustrative Example

Example-1:

A manufacturer produces three products A, B, C that he sells in the market. Annual sales volumes are indicated as follows:

Market	Products		
	A	B	C
I	8000	10000	15000
II	10000	2000	20000

(i) If unit sale prices of A, B and C are \$2.25, \$1.50 and 1.25 respectively, find the total revenue in each market with the help of matrices.

(ii) If the unit costs of the above three products are \$1.60, \$1.20 and \$0.90 respectively, find the gross profit with the help of matrices.

Solution:

(i) The total revenue in each market is given by the product matrix:

$$\begin{pmatrix} 2.25 & 1.50 & 1.25 \end{pmatrix} \begin{pmatrix} 8000 & 10000 \\ 10000 & 2000 \\ 15000 & 20000 \end{pmatrix}$$

$$= [51750 \quad 50500]$$

The total revenue from the market I is \$51750 and the total revenue from the market II is \$50500.

(ii) The total cost of products with the manufacturer sells in the markets are:

$$\begin{pmatrix} 1.60 & 1.20 & 0.90 \end{pmatrix} \begin{pmatrix} 8000 & 10000 \\ 10000 & 2000 \\ 15000 & 20000 \end{pmatrix}$$

$$= [38300 \quad 36400]$$

The total cost of products that the manufacturer sells in the market I and II are \$38300 and \$36400 respectively.

Required gross profit = (Total revenue received from both the markets) – (Total cost of products that the manufacturer sold in both the market)

$= (51750 + 50500) - (38300 + 36400)$

$= 102250 - 74700$

$= 27550.$

Example-2:

A finance company has offices located in every division, every district and every thana. Assume that there are five divisions, thirty districts and two hundred thanas. Each office has one head clerk, one cashier, one clerk and one peon. A divisional office has in addition one office superintendent, two clerks, one typist and one peon. A district office has in addition one clerk and one peon. The basic monthly salaries are as follows: office superintendent \$500, head clerk \$200, cashier \$175 clerks and typists \$150 and peon \$100. Using matrix notation, find the following

(i) The total number of posts of each kind in all the offices taken together.

(ii) The total basic monthly salary bill of each kind of office and

(iii) The total basic monthly salary bill of all the offices taken together.

Solution:

Let the number of offices can be arranged as elements of a row matrix

$$A = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix}$$

The composition of staff in various offices can be arranged in a 3×6 matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The basic monthly salaries of various types of employees of these offices correspond to the elements of the column matrix, $C = \begin{pmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{pmatrix}$

(i) Total numbers of posts of each kind in all the offices are the elements of the product matrix AB.

$$AB = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= [5 \quad 235 \quad 235 \quad 275 \quad 5 \quad 270]$$

Thus, the required numbers of posts in all the offices taken together are 5-office superintendent, 235 head clerks, 235 cashiers, 275 clerks, 5 typists and 270 peons.

(ii) $$\begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{pmatrix} = \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix}$$

Thus, the total basic monthly salary bill of each divisional, district and Thana offices are \$1675, \$875 and \$625 respectively.

(iii) Total basic monthly salary bill of all the offices is the element of the product matrix ABC,

$$\text{i.e., } ABC = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix} \times \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix} = 159625.$$

Thus, the total basic monthly salary bill of all the offices taken together is \$159625.

Example-3:

Three persons *A*, *B* and *C* posses Tk.3000, Tk.2000 and Tk.2500 respectively. *A* with his entire amount purchased 5 shares of Tk.*X* each, 3 shares of Tk.*Y* each and 4 shares of Tk.*Z* each. *B* purchased 3 shares of Tk.*X* each, 4 shares of Tk.*Y* each and 2 shares of Tk.*Z* each with his entire amount and *C* purchased 4 shares of Tk.*X* each, 3 shares of Tk.*Y* each and 4 shares of Tk.*Z* each with his entire amount. Determine the value of each share of different types.

Solution:

We have, $5x + 3y + 4z = 3000$

$$3x + 4y + 2z = 2000$$

$$4x + 3y + 4z = 2500$$

$$D = \begin{vmatrix} 5 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 10$$

$$D_x = \begin{vmatrix} 3000 & 3 & 4 \\ 2000 & 4 & 2 \\ 2500 & 3 & 4 \end{vmatrix} = 5000$$

$$D_y = \begin{vmatrix} 5 & 3000 & 4 \\ 3 & 2000 & 2 \\ 4 & 2500 & 4 \end{vmatrix} = 1000$$

$$D_z = \begin{vmatrix} 5 & 3 & 3000 \\ 3 & 4 & 2000 \\ 4 & 3 & 2500 \end{vmatrix} = 500$$

From Cramer's rule we know that,

We know from the Cramer's Rule, $\dfrac{x}{D_x} = \dfrac{y}{D_y} = \dfrac{z}{D_z} = \dfrac{1}{D}$

Hence $x = \dfrac{D_x}{D} = \dfrac{5000}{10} = 500$

$y = \dfrac{D_y}{D} = \dfrac{1000}{10} = 100$

$z = \dfrac{D_z}{D} = \dfrac{500}{10} = 50.$

Example-4:

To control a certain crop disease it is necessary to use 7 units of chemical *A*, 10 units of chemical *B* and 6 units of chemical *C*. One barrel of spray *P* contains 1 unit of *A*, 4 units of *B* and 2 units of *C*. One barrel of spray *Q* contains 3 units of *A*, 2 units of *B*, and 2 units of *C*. One barrel of spray *R* contains 4 units of *A*, 3 units of *B* and 2 units of *C*. How much of each type of spray should be used to control the disease?

Solution:

Let x barrels of spray *P*, y barrels of spray *Q* and z barrels of spray *R* be used to control the disease. Then we can write,

$$x + 3y + 4z = 7$$
$$4x + 2y + 3z = 10$$
$$2x + 2y + 2z = 6$$

The given information can be written under the matrix form as follows:

$$\begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

The determinant of the matrix A is, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{vmatrix} = 8$

The co-factor matrix of A is, $A^C = \begin{bmatrix} -2 & -2 & 4 \\ 2 & -6 & 4 \\ 1 & 13 & -10 \end{bmatrix}$

The Adjoint matrix of A is, $A^J = \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^J = \frac{1}{8} \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Hence $1\frac{1}{2}$ barrels of the spray P, $\frac{1}{2}$ barrel of spray Q and 1 barrel of spray R should be used to control the disease.

Example-5:

The cost of manufacturing the three types of motorcars is given below:

Car	Labor hours	Material used	Subcontracted works
A	40	100	50
B	80	150	80
C	100	250	100

Labor cost \$2 per hour, per unit material cost is \$0.5 and one unit of subcontracted work costs \$1. Find the total cost of manufacturing 3000, 2000 and 1000 vehicles of type A, B, C respectively. If the selling prices of car A, B, C are \$2000, 3500 and \$4500 respectively, then find the profit from selling those cars.

Solution:

Consider the following matrices,

$$M = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix} \qquad N = \begin{bmatrix} 2 \\ 10 \\ 1 \end{bmatrix}$$

$$MN = \begin{bmatrix} 180 \\ 315 \\ 425 \end{bmatrix}$$

This column matrix represents cost of each car A, B, C in that order.

Let $P = (3000 \quad 2000 \quad 1000)$, this row matrix represents number of cars A, B, C to be manufactured in that order.

Now $PMN = (1595000)$

Thus total cost of manufacturing three cars A, B, C is \$1595000.

Let $Q = \begin{bmatrix} 2000 \\ 3500 \\ 4500 \end{bmatrix}$; this column matrix represents the selling price of A, B, C.

$$\text{Now, Total Revenue} = PQ = (3000 \quad 2000 \quad 1000) \begin{bmatrix} 2000 \\ 3500 \\ 4500 \end{bmatrix}$$

$$= 17500000$$

Profit = Total revenue – Total cost = 17500000 – 1595000 = 15905000.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. *A*, *B* and *C* has Tk.480, Tk.760 and Tk.710 respectively. They utilized the amounts to purchase three types of shares of prices x, y and z respectively. *A* purchases 2 share of price x, 5 of price y and 3 of price z. *B* purchases 4 shares of price x, 3 of price y and 6 of price z. *C* purchases 1 share of price x, 4 of price y and 10 of price z. Find the value of x, y and z.

2. A manufacturing unit produces three types of television sets *A*, *B*, *C*. The following matrix shows the sale of television sets in two different cities.

$$\begin{pmatrix} A & B & C \\ 400 & 300 & 200 \\ 300 & 200 & 100 \end{pmatrix}$$

   If cost price of each set *A*, *B*, *C* is Tk.1000, Tk.2000, and Tk.3000 respectively and selling prices are Tk.1500, Tk.3000, Tk.4000 respectively, find the total profit using matrix algebra only.

3. The following matrix represents the results of the examination of MBA.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

   The rows represent the three sections of the class. The first three columns represent the number of students securing 1^{st}, 2^{nd} and 3^{rd} divisions respectively in that order and fourth column represents the number of students who failed in the examination.

   (i) How many students passed in three sections respectively?
   (ii) How many students failed in three sections respectively?
   (iii) Write down the matrix in which number of successful students is shown.
   (iv) Write down the column matrix where only failed students are shown.
   (v) Write down the column matrix showing students in the 1^{st} division from three sections.

4. A publishing house has two branches. In each branch, there are three offices. In each office, there are 3 peons, 4 clerks and 5 typists. In one office of a branch, 6 salesmen are also working. In each office of other branch 2 head clerks are also working. Using matrix notation find
   (i) the total number of posts of each kind in all the offices taken together in each branch.
   (ii) the total number of posts of each kind in all the offices taken together from both the branches.