Integral Calculus

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This unit is designed to introduce the learners to the basic concepts associated with Integral Calculus. Integral calculus can be classified and discussed into two threads. One is ***Indefinite Integral*** and the other one is ***Definite Integral***. The learners will learn about indefinite integral, methods of integration, definite integral and application of integral calculus in business and economics.

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Lesson-1: Indefinite Integral

After completing this lesson, you should be able to:

- ➢ Describe the concept of integration;
- ➢ Determine the indefinite integral of a given function.

Introduction

The process of differentiation is used for finding derivatives and differentials of functions. On the other hand, the process of integration is used (i) for finding the limit of the sum of an infinite number of infinitesimals that are in the differential form $f'(x)dx$ (ii) for finding functions whose derivatives or differentials are given, i.e., for finding anti-derivatives. Thus, reversing the process of differentiation and finding the original function from the derivative is called integration or anti-differentiation.

Reversing the process of differentiation and finding the original function from the derivative is called integration.

The integral calculus is used to find the areas, probabilities and to solve equations involving derivatives. Integration is also used to determine a function whose rate of change is known.

In integration whether the object be summation or anti-differentiation, the sign ∫, an elongated *S*, the first letter of the word 'sum' is most generally used to indicate the process of the summation or integration. Therefore, $\int f(x)dx$ is read the integral of *f (x)* with respect to *x*.

$\int f(x)dx$ is read the integral of f (x) with respect to x.

Again, integration is defined as the inverse process of differentiation.

Thus if $\frac{d}{dx}g(x) = f(x)$

then $\int f(x)dx = g(x) + c$

where *c* is called the constant of integration. Of course *c* could have any value and thus integral of a function is not unique! But we could say one thing here that any two integrals of the same function differ by a constant. Since *c* could also have the value zero, $g(x)$ is one of the values of $\int f(x)dx$. As *c* is unknown and indefinite, hence it is also referred to as Indefinite Integral.

Some Properties of Integration

The following two rules are useful in reducing differentiable expressions to standard forms.

(i) The integral of any algebraic sum of differential expression equals the algebraic sum of the integrals of these expressions taken separately.

i.e. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(ii) A constant multiplicative term may be written either before or after the integral sign.

i.e. $\int cf(x)dx = c\int f(x)dx$; where c is a constant.

Some Standard Results of integration

A list of some standard results by using the derivative of some well-known functions is given below:

(i) $\int dx = x + c \qquad \therefore \quad \frac{d}{dx}(x) = 1$

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad \therefore \quad \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \quad n \neq -1$

(iii) $\int \frac{1}{x} dx = \log x + c \qquad \therefore \quad \frac{d}{dx}(\log x) = \frac{1}{x}$

(iv) $\int e^x dx = e^x + c \qquad \therefore \quad \frac{d}{dx}(e^x) = e^x$

(v) $\int a^x dx = \frac{a^x}{\log a} + c \qquad \therefore \quad \frac{d}{dx}(a^x) = a^x \log a$

(vi) $\int \sin x\, dx = -\cos x + c \qquad \therefore \quad \frac{d}{dx}(-\cos x) = \sin x$

(vii) $\int \cos x\, dx = \sin x + c \qquad \therefore \quad \frac{d}{dx}(\sin x) = \cos x$

(viii) $\int \sec^2 x\, dx = \tan x + c \qquad \therefore \quad \frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\int \cos ec^2 x\, dx = -\cot x + c \qquad \therefore \quad \frac{d}{dx}(-\cot x) = \cos ec^2 x$

(x) $\int \sec x \tan x\, dx = \sec x + c \qquad \therefore \quad \frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\int \cos ecx \cot x\, dx = -\cos ecx + c$

$\therefore \quad \frac{d}{dx}(-\cos ecx) = \cos ex \cot x$

(xii) $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + c \qquad \therefore \quad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$

(xii) $\int \frac{1}{1 + x^2} dx = \tan^{-1} x + c \qquad \therefore \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$

(xiii) $\int \frac{1}{x\sqrt{x^2 - 1}} dx = \sec^{-1} x + c \qquad \therefore \quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1 + x^2}}$

(xiv) $\int \tan x\, dx = \log \sec x + c \qquad \therefore \quad \frac{d}{dx}(\log \sin x) = \cot x$

(xv) $\int \cot x\, dx = \log \sin x + c \qquad \therefore \quad \frac{d}{dx}(\log \sec x) = \tan x$

Illustrative Examples:

Example -1:

Evaluate $\int x^2 \, dx$

Solution:

$$\int x^2 \, dx = \frac{x^3}{3} + c$$

Example-2:

Evaluate $\int 5x \, dx$

Solution:

$$\int 5x \, dx = \frac{5}{2} x^2 + c$$

Example-3:

Evaluate $\int -2 \, dx$

Solution:

$$\int -2 \, dx = -2x + c$$

Example-4:

Evaluate $\int (2x + 3) \, dx$

Solution:

$$\int (2x+3) \, dx = \int 2x dx + \int 3 dx$$

$$= x^2 + 3x + c$$

Example-5:

Evaluate $\int (4x^2 - 7x + 6) \, dx$

Solution:

$$\int (4x^2 - 7x + 6) \, dx = \int 4x^2 dx - \int 7x dx + \int 6 dx$$

$$= \frac{4}{3} x^3 - \frac{7}{2} x^2 + 6x + c$$

Example-6:

Evaluate $\int \left(\frac{\mathrm{a}}{x^3} + \frac{b}{x^2} + 6 \right) dx$

Solution:

$$\int \left(\frac{\mathrm{a}}{x^3} + \frac{b}{x^2} + 6 \right) dx = \int \frac{a}{x^3} dx + \int \frac{b}{x^2} dx + \int 6 dx$$

$$= -\frac{a}{2x^2} - \frac{b}{x} + 6x + c$$

Example-7:

Evaluate $\int \frac{dx}{\sqrt{x-1} + \sqrt{x+3}}$

Solution:

$$\int \frac{dx}{\sqrt{x-1} + \sqrt{x+3}} = \int \frac{(\sqrt{x-1} - \sqrt{x+3})\, dx}{(\sqrt{x-1} + \sqrt{x+3})(\sqrt{x-1} - \sqrt{x+3})}$$

$$= \int \frac{(\sqrt{x-1} - \sqrt{x+3})\, dx}{x-1-x-3}$$

$$= \frac{-1}{4}\int \sqrt{x-1}\, dx + \frac{1}{4}\int \sqrt{x+3}\, dx$$

$$= \frac{1}{6}(x+3)^{\frac{3}{2}} - \frac{1}{6}(x-1)^{\frac{3}{2}} + c$$

Example-8:

Evaluate $\int (5e^x - x^{-2} + \frac{3}{x})\, dx; \quad x \neq 0$

Solution:

$$\int (5e^x - x^{-2} + \frac{3}{x})\, dx = \int 5e^x dx - \int x^{-2} dx + \int \frac{3}{x} dx$$

$$= 5e^x + \frac{1}{x} + 3\log x + c$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Integrate the following functions w. r. t. x

   (i) $\int (5x^3 + \frac{6}{x^3})dx$

   (ii) $\int \frac{1}{\sqrt{x}} dx$

   (iii) $\int (\sqrt{x} - \frac{1}{\sqrt{x}})dx$

   (iv) $\int (x^3 + \frac{1}{x^3})dx$

   (v) $\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}$

   (vi) $\int (2e^x + 7^x + \frac{1}{\sqrt{x}})dx$

Lesson-2: Methods of Integration

After studying this lesson, you should be able to:

- Describe the methods of integration;
- Determine the integral of any function.

Introduction

In comparing integral and differential calculus, most of the mathematicians would agree that the integration of functions is a more complicated process than the differentiation of functions. Functions can be differentiated through application of a number of relatively straightforward rules. This is not true in determining the integrals of functions. Integration is much less straightforward and often requires considerable ingenuity.

Certain functions can be integrated quite simply by applying rules of integration.

Certain functions can be integrated quite simply by applying rules of integration. A natural question is that what happens when rules of integration cannot be applied directly. Such functions require more complicated techniques. This lesson discusses four techniques that can be employed when the other rules do not apply and when the structure of the integrand is of an appropriate form. In general, experience is the best guide for suggesting the quickest and simplest method for integrating any given function.

Methods of Integration

The following are the four principal methods of integration:

(i) Integration by substitution;
(ii) Integration by parts;
(iii) Integration by successive reductions;
(iv) Integration by partial fraction.

Integration by Substitution

Substituting a new suitable variable for the given independent variable and integrating with respect to the substituted variable can often facilitate Integration.

Substituting a new suitable variable for the given independent variable and integrating with respect to the substituted variable can often facilitate Integration. Experience is the best guide as to what substitution is likely to transform the given expression into another that is more readily integrable. In fact this is done only for convenience.

The following examples will make the process clear.

Example-1:

Evaluate $\int (ax+b)^5 dx$

Solution:

Let, $ax + b = z$

$adx = dz$

$dx = \frac{1}{a} dz$

$$\int (ax+b)^5 dx = \int z^5 . \frac{1}{a} dz$$

$$= \frac{1}{a}\int z^5 dz$$

$$= \frac{1}{a}.\frac{z^6}{6} + c$$

$$= \frac{1}{a}.\frac{(ax+b)^6}{6} + c$$

Example-2:

Evaluate $\int x(x^2+4)^5 dx$

Solution:

Let, $x^2 + 4 = z$

$$2xdx = dz$$

$$xdx = \frac{1}{2}dz$$

$$\int x(x^2+4)^5 dx = \int z^5.\frac{1}{2}dz$$

$$= \frac{1}{2}\int z^5 dz$$

$$= \frac{1}{2}.\frac{z^6}{6} + c$$

$$= \frac{1}{12}(x^2+4)^6 + c$$

Example-3:

Evaluate $\int x^2 e^{x^3} dx$

Solution:

Let, $x^3 = z$

$$3x^2 dx = dz$$

$$x^2 dx = \frac{1}{3}dz$$

$$\int x^2 e^{x^3} dx = \int e^z.\frac{1}{3}dz$$

$$= \frac{1}{3}e^z + c$$

$$= \frac{1}{3}e^{x^3} + c$$

Example-4:

Evaluate $\int x\sqrt{x^2+1}\, dx$.

Solution:

Let, $x^2 + 1 = z$

$$2xdx = dz$$

$$xdx = \frac{1}{2}dz$$

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{z}.\ \frac{1}{2}dz$$

$$= \frac{1}{2}\int z^{\frac{1}{2}}dz$$

$$= \frac{1}{2}.\frac{z^{\frac{3}{2}}}{\frac{3}{2}} + \text{c}$$

$$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$$

Integration by Parts

Integration by parts is a special method that can be applied in finding the integrals of a product of two integrable functions.

Integration by parts is a special method that can be applied in finding the integrals of a product of two integrable functions. This method of integration is derived from the rule of differentiation of a product of two functions.

If u and v are two functions of x then,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x, we get

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = \int uv - \int v\frac{du}{dx}dx$$

Putting u = f(x), $\frac{\text{dv}}{\text{dx}}$ = g(x) then v = $\int$ g(x)dx

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx$$

Thus integral of the product of two functions

= 1st function × integral of the 2nd – integral of (differential of 1st × integral of 2nd).

It is clear from the formula that it is helpful only when we know integral of at least one of the two given functions.

The following examples will illustrate how to apply this rule.

Example-5:

Evaluate $\int xe^x dx$

Solution:

$$\int xe^x dx = x\int e^x dx - \int \{\frac{d}{dx}(x)\int e^x dx\}dx$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c$$

Example-6:

Evaluate $\int \log x \, dx$

Solution:

$$\int \log x \, dx = \int \log x.1.\, dx$$

$$= \log x\int 1.dx - \int \{\frac{d}{dx}(\log x)\int 1.dx\}dx$$

$$= \log x.x - \int \frac{1}{x}.xdx$$

$$= x\log x - \int dx$$

$$= x\log x - x + c\,.$$

Example 7:

Evaluate $\int x^2 \log x \, dx$

Solution:

$$\int x^2 \log x \, dx = \log x\int x^2 dx - \int \{\frac{d}{dx}(\log x)\int x^2 dx\}dx$$

$$= \log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}.dx$$

$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}dx$$

$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

Integration by Successive Reduction

Any formula expressing a given integral in terms of another that is simpler than it, is called a reduction formula for the given integral. In practice, however, the reduction formula for a given integral means that the integral belongs to class of integrals such that it can be expressed in terms of one or more integrals or lower orders belonging to the same class; by successive application of the formula, we arrive at integrals which can be easily integrated and hence the given integral can be evaluated.

Any formula expressing a given integral in terms of another that is simpler than it, is called a reduction formula for the given integral.

Example-8:

Evaluate $\int x^3 e^{3x} \, dx$

Solution:

$$\int x^3 e^{3x}\,dx = x^3\int e^{3x}dx - \int\{\frac{d}{dx}(x^3)\int e^{3x}dx\}dx$$

$$= \frac{x^3e^{3x}}{3} - \int x^2e^{3x}dx$$

$$= \frac{x^3e^{3x}}{3} - \left[\frac{x^2e^{3x}}{3} - \frac{2}{3}\int xe^{3x}dx\right]$$

$$= \frac{x^3e^{3x}}{3} - \frac{x^2e^{3x}}{3} + \frac{2}{3}\left[\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx\right]$$

$$= \frac{x^3e^{3x}}{3} - \frac{x^2e^{3x}}{3} + \frac{2}{9}xe^{3x} - \frac{2}{27}e^{3x} + c$$

Integration by Partial Fraction

Many rational functions exist which cannot be integrated by the rules of integration.

Rational functions have the form of a quotient of two polynomials. Many rational functions exist which cannot be integrated by the rules of integration presented earlier. When these occur, one possibility is that the rational function can be restated in an equivalent form consisting of more elementary functions and then each of the component fractions can be easily integrated separately. The following examples illustrate the decomposition of a rational function into equivalent partial fractions.

Example-9:

Evaluate $\int \frac{x+3}{x^2+3x+2}dx$

Solution:

$$\int \frac{x+3}{x^2+3x+2}dx = \int \frac{x+3}{(x+1)(x+2)}dx$$

$$= \int\left(\frac{2}{x+1} - \frac{1}{x+2}\right)dx$$

$$= 2\log(x+1) - \log(x+2) + c$$

Example-10:

Evaluate $\int \frac{3x+1}{x+1}dx$

Solution:

$$\int \frac{3x+1}{x+1}dx = \int(3 - \frac{2}{x+1})\,dx$$

$$= \int 3\,dx - \int \frac{2}{x+1}dx$$

$$= 3x - 2\log(x+1) + c$$

Example-11:

Evaluate $\int \frac{5x+8}{x^2+4x+4}\,dx$

Solution:

$$\int \frac{5x+8}{x^2+4x+4}\,dx = \int\left(\frac{5}{x+2} - \frac{2}{(x+2)^2}\right)dx$$

$$= \int \frac{5}{x+2}\,dx - \int \frac{2}{(x+2)^2}\,dx$$

$$= 5\log(x+2) + \frac{2}{x+2} + c$$

Example 12:

Evaluate $\int \frac{x^3-2x}{x-1}\,dx$

Solution:

$$\int \frac{x^3-2x}{x-1}\,dx = \int (x^2+x-1-\frac{1}{x-1})\,dx$$

$$= \int (x^2+x-1)dx - \int \frac{1}{x-1}dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} - x - \log(x-1) + c$$

Example 13:

Evaluate $\int \frac{2x^2-1}{x^3+x^2}\,dx$

Solution:

$$\int \frac{2x^2-1}{x^3+x^2}\,dx = \int\left(\frac{x-1}{x^2} + \frac{1}{x+1}\right)\mathrm{dx}$$

$$= \int \frac{x-1}{x^2}dx + \int \frac{1}{x+1}dx$$

$$= \log x + \frac{1}{x} + \log(x+1) + c$$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Integrate the following functions w. r. t. x

(i) $\int \frac{dx}{x^2(a+bx)^2}$

(ii) $\int \frac{8x^2}{(x^3+2)^3}dx$

(iii) $\int x \log x dx$

(iv) $\int x^3 e^x dx$

(v) $\int \frac{x^2-2}{(x+1)(x^2+1)}dx$

Lesson-3: Definite Integral

After studying this lesson, you should be able to:

- Describe the concept of definite integral;
- Evaluate definite integrals.

Introduction

In Geometry and other application areas of integral calculus, it becomes necessary to find the difference in the values of an integral *f(x)* for two assigned values of the independent variable *x*, say, *a, b, (a < b),* where *a* and *b* are two real numbers. The difference is called the definite integral of *f(x)* over the domain (*a, b*) and is denoted by

$$\int_a^b f(x)dx \ldots\ldots\ldots\ldots(i)$$

If *g(x)* is an integral of *f(x)*, then we can write,

$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a) \ldots\ldots\ldots\ldots\ldots(ii)$$

Here $\int_a^b f(x)dx$ is called the definite integral, as the constant of integration does not appear in it. If we consider [*g(x)* + *c*] instead of *g(x)*, we have

$$\int_a^b f(x)dx = \left[g(x)+c\right]_a^b = g(b) + c - g(a) - c = g(b) - g(a) \ldots\ldots\ldots(iii)$$

Thus, from (ii) or (iii), we get a specific numerical value, free of the variable *x* as well as the arbitrary constant *c*. This value is called the definite integral of *f(x)* from *a* to *b*. We refer to *a* as the lower limit of integration and to *b* as the upper limit of integration.

Properties of Definite Integral

Some important properties of definite integral are given below:

$$(i)\int_a^b f(x)dx = \int_b^a f(z)dz$$

$$(ii)\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(iii)\int_a^a f(x)dx = 0$$

$$(iv)\int_a^c f(x)dx = \int_a^b f(x) + \int_b^c f(x)dx$$

$$(v)\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$(vi)\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$(vii)\int_0^{na} f(x)dx = n\int_0^a f(x)dx \qquad if \quad f(a+x) = f(x)$$

$$(viii)\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \qquad if \quad f(2a-x) = f(x)$$

$$(ix)\int_{-a}^{+a} f(x)dx = 2\int_a^a f(x)dx$$

Illustrative Examples:

Example-1:

Evaluate $\int_a^b c\,\mathrm{dx}$

Solution:

$$\int_a^b c\,\mathrm{dx} = c\int_a^b \mathrm{dx} = c[x]_a^b = \mathrm{c\,(b - a)}$$

Example-2:

Evaluate $\int_1^4 x^2\,\mathrm{dx}$

Solution:

$$\int_1^4 x^2\,\mathrm{dx} = \left[\frac{x^3}{3}\right]_1^4 = 21$$

Example-3:

Evaluate $\int_1^9 \frac{dx}{\sqrt{x}}\,dx$

Solution: $\int_1^9 \frac{dx}{\sqrt{x}}\,dx = \int_1^9 x^{-\frac{1}{2}}\,dx = 2\left[x^{\frac{1}{2}}\right]_1^9 = 4$

Example-4:

Evaluate $\int_1^4 (2\mathrm{x}^2 - 4\mathrm{x} + 5)\,\mathrm{dx}$

Solution:

$$\int_1^4 (2\mathrm{x}^2 - 4\mathrm{x} + 5)\,\mathrm{dx} = \left[\frac{2x^3}{3} - 2x^2 + 5x\right]_1^4 = 27$$

Example-5:

Evaluate $\int_{2}^{4} \frac{x\,dx}{x^2-1}$

Solution:

Let $x^2 - 1 = z$

$2xdx = dz$

when $x = 2$, then $z = 3$

when $x = 4$, then $z = 15$

$$\int_{2}^{4} \frac{x\,dx}{x^2-1} = \int_{3}^{15} \frac{dz}{2z} = \frac{1}{2}\left[\log z\right]_{3}^{15} = \frac{1}{2}(\log 15 - \log 3)$$

Example-6:

Evaluate $\int_{0}^{4} x\sqrt{x^2+9}\,dx$

Solution:

Let $x^2 + 9 = z$

$2xdx = dz$

when $x = 0$, then $z = 9$

when $x = 4$, then $z = 25$

$$\int_{0}^{4} x\sqrt{x^2+9}\,dx = \frac{1}{2}\int_{9}^{25} \sqrt{z}\,dz = \frac{1}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_{9}^{25} = \frac{98}{3}$$

Example-7:

Evaluate $\int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx$

Solution:

Let $I = \int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx = \int_{0}^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}\,dx = \int_{0}^{\pi} \frac{(\pi - x)dx}{1+\sin x}\,dx$

$$= \int_{0}^{\pi} \frac{\pi dx}{1+\sin x}\,dx - \int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx$$

$$2I = \pi\int_{0}^{\pi} \frac{dx}{1+\sin x}\,dx = \pi\int_{0}^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx =$$

$$\pi\left[\int_{0}^{\pi} \sec^2 x\,dx - \int_{0}^{\pi} \tan x \sec x dx\right]$$

$$= \pi\left[\tan x - \sec x\right]_{0}^{\pi}$$

$$= 2\pi$$

$\therefore I = \pi$.

Proper and Improper Integrals

An integral is said to be proper integral when it is bounded and the range of integration is finite.

Proper Integrals: An integral is said to be proper integral when it is bounded and the range of integration is finite. For example, $\int_0^2 f(x)dx$, $\int_1^2 f(x)dx$ etc.

Improper Integrals: When the range of integration is finite but the integrand is unbounded for some values in the range of integration, then it is called the improper integral of first kind. e.g. $\int_1^2 \frac{dx}{(1-x)(2-x)}$, $\int_0^1 \frac{dx}{x^2}$ etc.

When the range of integration is infinite but the integrand is bound, then it is called improper integrals of second kind. e.g., $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, $\int_{-\infty}^{\infty} f(x)dx$ etc.

These types of improper integrals are determined as if

$$\int_a^{\infty} f(x)dx = \lim_{w\to\infty}\int_a^{w} f(x)dx.$$

When the limit exists, the integral is said to be convergent to that limit and when the limit does not exist, the integral is said to be divergent to that limit.

When the limit exists, the integral is said to be convergent to that limit and when the limit does not exist, the integral is said to be divergent to that limit. If, however, the limit does not converge or diverge then it is said to be oscillatory.

Example-8:

Determine whether the improper integral $\int_0^{\infty} \frac{e^x}{2}dx$ is convergent or divergent.

Solution:

$$\int_0^{\infty} \frac{e^x}{2}dx = \lim_{a\to\infty}\int_0^{a} \frac{e^x}{2}dx = \lim_{a\to\infty}\frac{1}{2}(e^a - 1) = \infty$$

Thus the given improper integral is divergent.

Example-9:

Determine whether the improper integral $\int_{-\infty}^{0} e^x dx$ is convergent or divergent.

Solution:

$$\int_{-\infty}^{0} e^x dx = \lim_{a \to -\propto} \int_{a}^{0} e^x dx = \lim_{a \to -\propto} (e^0 - e^a)$$

$$= \lim_{a \to -\propto} (1 - e^a) = 1 - 0 = 1$$

Thus the given improper integral is convergent and its value is 1.

Multiple Integrals

Integration of a function in one variable generates an area (a surface) from a line. A function in two variables generates a volume from a surface. Because a function in two variables describes a surface with different curvature in each direction, both variables are responsible for generating the appropriate volume. To find the volume, the integral must be taken with respect to both variables.

A function in two variables generates a volume from a surface.

Example-10:

Find the value of $\int_0^1 \int_0^2 (x+2)dydx$

Solution:

$$\int_0^1 \int_0^2 (x+2)dydx = \int_0^1 [\int_0^2 (x+2)dy]dx$$

$$= \int_0^1 [xy + 2y]_0^2 dx$$

$$= \int_0^1 [2x + 4]dx$$

$$= [x^2 + 4x]_0^1$$

$$= 5$$

Example-11:

Find the value of $\int_2^3 \int_1^2 \int_2^5 xydzdydx$

Solution:

$$\int_2^3 \int_1^2 \int_2^5 xydzdydx = \int_2^3 \int_1^2 [\int_2^5 xydz]dydx$$

$$= \int_2^3 \int_1^2 [\, xyz \,]_2^5 \, dydx$$

$$= 3\int_2^3 [\int_1^2 xy\, dy \,] dx$$

$$= 3\int_2^3 [\frac{1}{2} xy^2 \,]_1^2 \, dx$$

$$= 3.\frac{3}{2}.\int_2^3 x\, dx$$

$$= \frac{45}{4}$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Evaluate the following integrals:

   (i) $\int_a^b e^x \, dx$

   (ii) $\int_0^1 x^3 \sqrt{(1+3x^4)} \, dx$

   (iii) $\int_a^b \frac{\log x}{x} \, dx$

   (iv) $\int_0^\infty \frac{dx}{1+x^2}$

   (v) $\int_0^{\pi/2} \log \sin x \, dx$

   (vi) $\int_0^2 \int_0^x (x^2 + y^2) \, dy dx$

   (vii) $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2) \, dx dy dz$

Lesson-4: Applications of Integration in Business

After studying this lesson, you should be able to:

- Express the importance of integration in Business and Economics;
- Apply integration in different types of business decisions.

Introduction

Marginal cost is the change in total cost from an incremental change in output and only variable costs change with the level of output.

The knowledge of integration is widely used in business and economics. For example, net investment *I* is defined as the rate of change in capital stock formation *K* over time *t*. If the process of capital formation is continuous over time, *I(t)* = $\frac{dK(t)}{dt} = K'(t)$. From the rate of investment, the level of capital stock can be estimated. Capital stock is the integral with respect to time of net investment. Similarly the integral can be used to estimate total cost from the marginal cost. Since marginal cost is the change in total cost from an incremental change in output and only variable costs change with the level of output. Economic analysis that traces the time path of variables or attempts to determine whether variables will converge towards equilibrium over time is called dynamics. Thus, we can use integration in many business decision making processes. In this lesson, we discuss about a few sample applications of integration. The following examples illustrate sample applications of integral calculus.

Illustrative Examples:

Example-1:

Find the area bounded by the curve $y = x^2$, the *x*-axis and the lines $x = 1$ and $x = 3$.

Solution:

We know that, Area = $\int_a^b y\,dx = \int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{26}{3}$

Example-2:

The marginal cost function of a product is given by $\frac{dC}{dq} = 100 - 10q + 0.1q^2$, where q is the output. Find the total and average cost functions of the firm assuming that its fixed cost is \$1500.

Solution:

Given that $\frac{dC}{dq} = 100 - 10q + 0.1q^2$

Integrating this with respect to *q*, we get,

C = $\int (100 - 10q + 0.1q^2)\,dq$

C = $100q - 5q^2 + \frac{0.10}{3}q^3 + K$

Now the fixed cost is 1500; i.e., when $q = 0$, C = 1500.
$\therefore$ K = 1500

Hence the total cost function is, C = $100q - 5q^2 + \frac{0.10}{3}q^3 + 1500$

And the average cost is $\frac{C}{q} = 100 - 5q + \frac{0.1}{3}q^2 + \frac{1500}{q}$

Example-3:

The marginal revenue $R'(x) = 25 - 8x + 6x^2 + 4x^3$. Find the revenue function.

Solution:

The revenue function is

$$R(x) = \int R'(x)\,dx = \int (25 - 8x + 6x^2 + 4x^3)\,dx$$

$$= 25x - 4x^2 + 2x^3 + x^4 + K.$$

We know that when no product is sold then the revenue is zero. i.e., when $x = 0$, R = 0.

Thus, K = 0.

Thus the revenue function is $R(x) = 25x - 4x^2 + 2x^3 + x^4$.

Example-4:

The marginal cost function for a certain commodity is $MC = 3q^2 - 4q + 5$. Find the cost of producing the 11th through the 15th units, inclusive.

Solution: $\int_{11-1}^{15} (3q^2 - 4q + 5)dq$

$$= \int_{10}^{15} (3q^2 - 4q + 5)dq$$

$$= \left[q^3 - 2q^2 + 5q\right]_{10}^{15}$$

$$= 2150$$

Example-5:

A Company determines that the marginal cost of producing x units of a particular commodity during a one-day operation is $MC = 16x - 1591$, where the production cost is in dollar. The selling price of a commodity is fixed at $9 per unit and the fixed cost is $1800 per day.

(i) Find the cost function.
(ii) Find the revenue function.
(iii) Find the profit function.
(iv) Find the maximum profit that can be obtained in a one-day operation.

Solution: Given that $MC = 16x - 1591$
FC = 1800
P = 9

(i) Cost function, TC = $\int MC\, dx$

$= \int (16x - 1591)\, dx$

$= 8x^2 - 1591x + c$

$= 8x^2 - 1591x + 1800$

(ii) Revenue = $P \times x = 9x$

(iii) Profit = TR – TC

$= 9x - (8x^2 - 1591x + 1800)$

$= 1600x - 8x^2 - 1800$

(iv) Profit $y = 1600x - 8x^2 - 1800$

$\frac{dy}{dx} = 1600 - 16x$

For maximum or minimum, $\frac{dy}{dx} = 0$

$1600 - 16x = 0$

$x = 100$

Again, $\frac{d^2y}{dx^2} = -16$

Hence the required profit maximizing sales volume is x = 100.
Required maximum profit $y = 1600x - 8x^2 - 1800$

$= 1600(100) - 8(100)^2 - 1800$

= $78200.

Example-6:

After an advertising campaign a product has sales rate *f(t)* given by $f(t) = 1000\, e^{-0.5t}$ where *t* is the number of months since the close of the campaign.

(i) Find the total cumulative sales after 3 months.
(ii) Find the sales during the fourth month.
(iii) Find the total sale as a result of campaign.

Solution:

Let *F(t)* is the total sale after *t* months since the close of the campaign. The sale rate is *f(t)*.

$F(t) = \int_0^t f(t)dt$

(i) The total cumulative sales after 3 months, $F(3) = \int_0^3 f(t)dt$

$\int_0^3 f(t)dt = \int_0^3 1000e^{-0.5t} dt$

$$= \frac{1000}{-0.5}\left[e^{-0.5t}\right]_0^3$$
$$= -2000\ [e^{-1.5} - 1]$$
$$= -2000\ (0.2231 - 1)$$
$$= 1554 \text{ units.}$$

(ii) Sales during the fourth month = $\int_3^4 1000e^{-0.5t}dt$

$$\int_3^4 1000e^{-0.5t}dt = \frac{1000}{-0.5}\left[e^{-0.5t}\right]_3^4$$
$$= -2000\ [e^{-2.0} - e^{-1.5}]$$
$$= -2000\ (0.1353 - 2231)$$
$$= 175.6 \text{ units.}$$

(iii) Total sales as a result of campaign = $\int_0^\infty 1000e^{-0.5t}dt$

$$\int_0^\infty 1000e^{-0.5t}dt = \frac{1000}{-0.5}\left[e^{-0.5t}\right]_0^\infty$$
$$= -2000\ (0 - 1)$$
$$= 2000 \text{ units.}$$

Example-7:

If \$500 is deposited each year in a saving account pays 5.5 % per annum compounded continuously, how much is in the account after 4 years?

Solution:

Given that, payment per year, $P = 500$

Rate of interest, $r = 0.055$

Time, $t = 4$

Amount of the future value A = $\int_0^t P\, e^{rt} dt$

$$= \int_0^4 500e^{0.055t}dt$$
$$= \frac{500}{0.055}\left[e^{0.055t}\right]_0^4$$
$$= 9090.91\ [e^{0.22} - e^{0}]$$
$$= 9090.91(0.246076)$$
$$= \$2237.$$

Example-8:

If the marginal revenue and the marginal cost for an output x of a commodity are given as $MR = 5 - 4x + 3x^2$ and $MC = 3 + 2x$, and if the fixed cost is zero, find the profit function and the profit when the output $x = 4$.

Solution:

Given that, $MR = 5 - 4x + 3x^2$

$MC = 3 + 2x$

Profit = $TR - TC$

$= \int MRdx - \int MCdx$

$= \int (5 - 4x + 3x^2)dx - \int (3 + 2x)dx$

$= (5x - 2x^2 + x^3 + c_1) - (3x + x^2 + c_2)$

Since the fixed cost is zero so that $c_2 = 0$; for $x = 0$, total revenue = 0

Profit = $2x - 3x^2 + x^3$

When $x = 4$, the profit = $(2 \times 4) - 3(4)^2 + (4)^3 = 24$.

Example-9:

Find the total cost function if it is known that the cost of zero output is c and that marginal cost of output x is $ax + b$.

Solution:

We are given that, Marginal cost (MC) = $ax + b$.

$$\frac{dTC}{dx} = ax + b$$

$$TC = \int (ax + b)dx$$

$$TC = \frac{ax^2}{2} + bx + K$$

When $x = 0$, $TC = c$, so, $K = c$.

Hence the total cost function is given by, TC = $\frac{ax^2}{2} + bx + c$

Example-10:

Let the rate of net investment is given by $I(t) = 9t^{1/2}$, find the level of capital formation in (i) 8 years (ii) for the fifth through the eighth years.

Solution:

(i) K = $\int_0^8 9t^{1/2}dt = 6t^{3/2}\Big|_0^8 = 135.76$

(ii) K = $\int_4^8 9t^{1/2}dt = 6t^{3/2}\Big|_4^8 = 87.76$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Marginal cost is given by $MC = 25 + 30Q - 9Q^2$. Fixed cost is 55. Find the (i) total cost (ii) average cost, and (iii) variable cost functions.
2. Marginal revenue is given by $MR = 60 - 2Q - 2Q^2$. Find the total revenue function and the demand function.
3. The rate of net investment is $I = 40t^{3/5}$ and capital stock at $t = 0$ is 75. Find the capital function K.