Maxima and Minima

9

This unit is designed to introduce the learners to the basic concepts associated with Optimization. The readers will learn about different types of functions that are closely related to optimization problems. This unit discusses maxima and minima of simple polynomial functions and develops the concept of critical points along with the first derivative test and next the concavity test. This unit also discusses the procedure of determining the optimum (maximum or minimum) point with single variable function, multivariate function and constrained equations. Some relevant business and economic applications of maxima and minima are also provided in this unit for clear understanding to the learners.

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Lesson-1: Optimization of Single Variable Function

After studying this lesson, you should be able to:

- Describe the concept of different types of functions;
- Explain the maximum, minimum and point of inflection of a function;
- Describe the methodology for determining optimization conditions for mathematical functions with single variable;
- Determine the maximum and minimum of a function with single variable;
- Determine the inflection point of a function with single variable.

Introduction

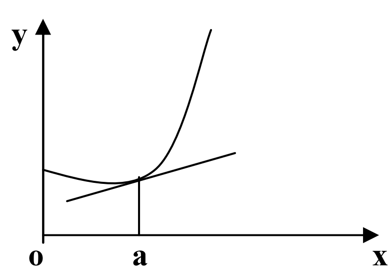
Optimization is a predominant theme in management and economic analysis. For this reason, the classical calculus methods of finding free and constrained extrema and the more recent techniques of mathematical programming occupy an important place in management and economics. The most common criterion of choice among alternatives in economics is the goal of maximizing something (i.e. profit maximizing, utility maximizing, growth rate maximizing etc.) or of minimizing something (i.e. cost minimizing). Economically, we may categorize such maximization and minimization problems under the general heading of optimization, which means 'the quest for the best'. The present lesson is devoted to a brief discussion of optimization with single variable function.

The most common criterion of choice among alternatives in economics is the goal of maximizing something or of minimizing something.

Increasing Function

A function $f(x)$ is said to be increasing at $x = a$ if the immediate vicinity of the point $(a, f(a))$ the graph of the function rises as it moves from left to right.

Since the first derivative measures the rate of change and slope of a function, a positive first derivative at $x = a$ indicates the function is increasing at $x=a$, i.e. $f'(a) > 0$ means increasing function at $x = a$.

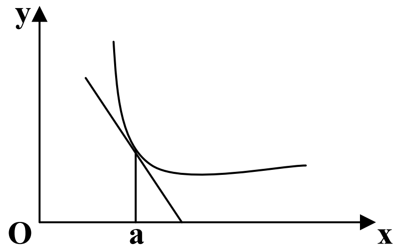


Decreasing Function:

A function $f(x)$ is said to be decreasing at $x = a$ if the immediate vicinity of the point $(a, f(a))$ the graph of the function falls as it moves from left to right. Since the first derivative measures the rate of change and slope

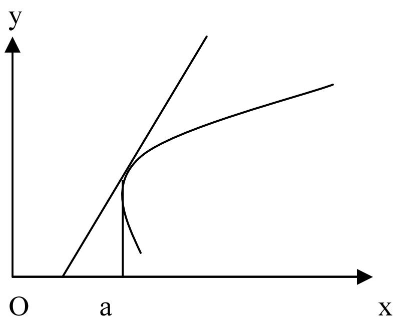
A function $f(x)$ is said to be decreasing at $x = a$ if the immediate vicinity of the point $(a, f(a))$ the graph of the function falls as it moves from left to right.

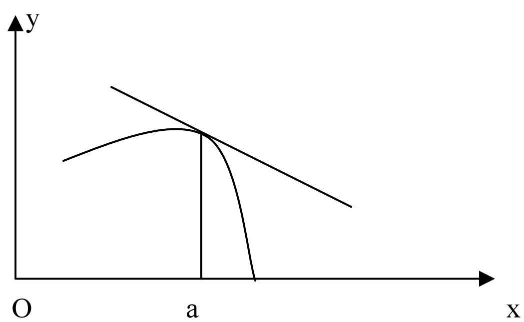
of a function, a negative first derivative at $x = a$ indicates the function is decreasing at $x = a$, i.e. $f'(a) < 0$; means decreasing function at $x = a$.



Concave Function:

A function $f(x)$ is concave at $x = a$ if in some small region close to the point $(a, f(a))$ the graph of the function lies completely below its tangent line. A negative second derivative at $x = a$ denotes the function is concave at $x = a$. The sign of the first derivative is immaterial for concavity.



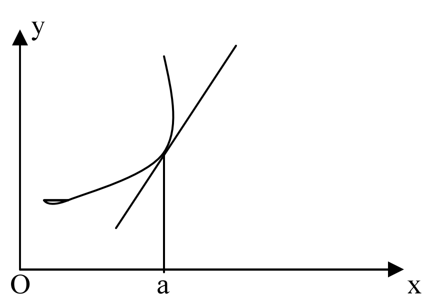


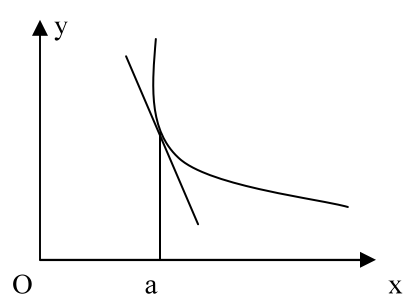
$f'(a) > 0$; $f''(a) < 0$ $\quad$ $f'(a) < 0$; $f''(a) < 0$

Convex Function:

A function f(a) is convex at x = a if in an area very close to the point (a, f(a)) the graph of the function lies completely above its tangent line.

A function $f(a)$ is convex at $x = a$ if in an area very close to the point $(a, f(a))$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$.





$f'(a) > 0$; $f''(a) > 0$ $\quad$ $f'(a) < 0$; $f''(a) > 0$

Inflection Points:

An inflection point is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa. Inflection points occur only where the second derivative equals zero or is undefined. Hence, the sign of the first derivative is immaterial.

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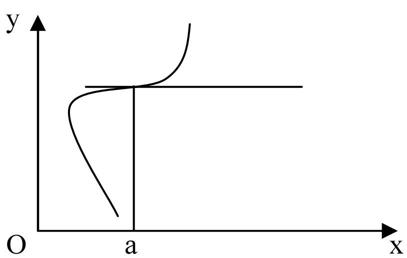
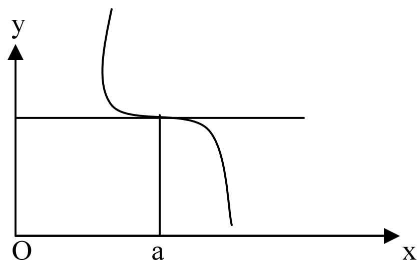
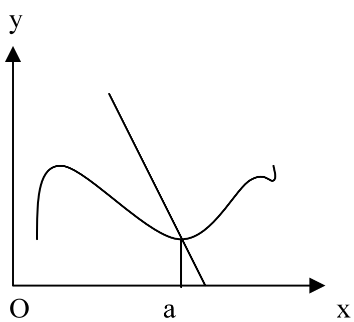


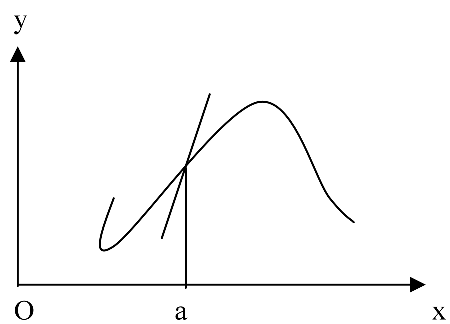
$f'(a) = 0;\ f''(a) = 0$



$f'(a) = 0; \quad f''(a) = 0$



$f'(a) < 0; \quad f''(a) = 0$



$f'(a) > 0; \quad f''(a) = 0$

Maxima:

A function $f(x)$ is said to have attained at any of its maximum values at $x = a$ if the function ceases to increase and begins to decrease at $x = a$.

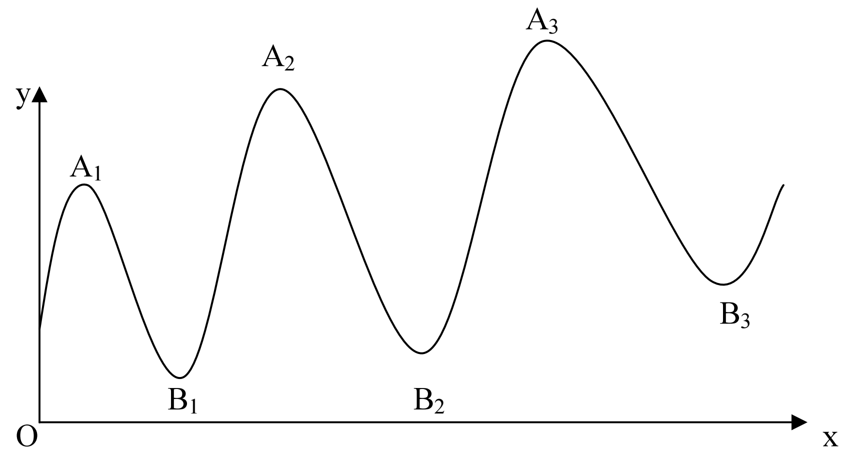
A function f (x) is said to have attained at any of its maximum values at x = a if the function ceases to increase and begins to decrease at x = a.

In other words, a function $f(x)$ is said to be maximum at a point $x = a$ if $f(a)$ is greater than any other values of $f(x)$ in the neighbourhood of $x = a$.

Minima:

A function $f(x)$ is said to have attained at any of its minimum values at $x = a$ if the function ceases to decrease and begins to increase at $x = a$.

In other words, a function $f(x)$ is said to be maximum at a point $x = a$ if $f(a)$ is less than any other values of $f(x)$ in the neighbourhood of $x = a$.



In the above figure, which represents graphically the function $y = f(x)$, a continuous function, has maximum values at A_1, A_2, A_3 and has minimum values at B_1, B_2, B_3.

From the figure, the following features regarding maxima and minima of a continuous function will be apparent:

(i) A function may have more than one maximum and minimum values. A function may have several maxima and minima in an interval where the function is defined.

(ii) It is not necessary that a maximum value of a function is always greater than other minimum values of the function. Any maximum value of a function may be less than any other minimum value of a function.

(iii) In between two maxima, there should be at least one minimum value of the function. Similarly, at least one maximum value of the function must lie between two minimum values of the function. In other words, there is a minimum value of the function between two consecutive maximum values and vice versa. Thus we observe that maximum and minimum values of a function occur alternately.

(iv) In Calculus, we are concerned with a relative maximum or a relative minimum value of a function and not with an absolute maximum or absolute minimum values.

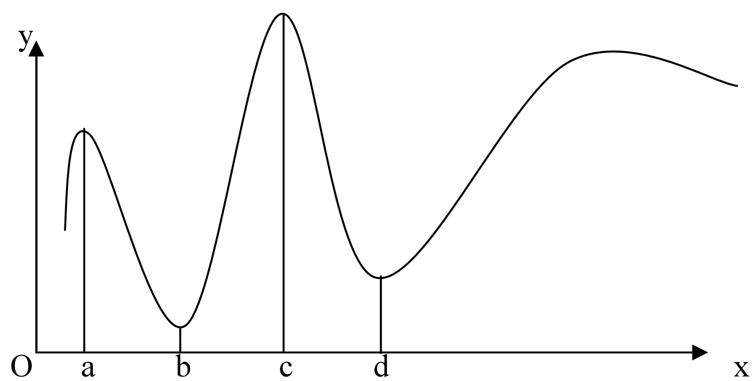
(v) The maximum or minimum point is called a turning point in a curve. The values of the function at these points are called turning values. For the maximum point, the curve ceases to ascend and begins to descend. Such turning value of the function is a maximum. For the minimum point, the curve ceases to descend and begins to ascend and the turning value is a minimum at the turning point.

A function $f(x)$ is said to reach an absolute maximum at $x = a$ if $f(a) > f(x)$ for any other values of x in the domain of $f(x)$.

Absolute and Local Maxima and Minima:

A function $f(x)$ is said to reach an absolute maximum at $x = a$ if $f(a) > f(x)$ for any other values of x in the domain of $f(x)$. On the other hand, a function *f (x)* is said to reach an absolute minimum at $x = a$ if $f(a) < f(x)$ for any other values of x in the domain of $f(x)$.

If a function $f(x)$ is defined on an interval (b, c) which contains $x = a$, $f(x)$ is said to reach a relative (local) maximum at $x = a$ if $f(a) \geq f(x)$ for all other values of x within the interval (b, c). A relative (local) maximum refers to a point where the value of $f(x)$ is greater than values for any other points that are nearby. Again, if a function $f(x)$ is defined on an interval (b, c) that contains $x = a$, $f(x)$ is said to reach a relative (local) minimum at $x = a$ if $f(a) \leq f(x)$ for all other values of x within the interval (b, c). A relative (local) minimum refers to a point where the value of $f(x)$ is lower than values for any other points that are nearby.



From the figure, $f(x)$ reaches an absolute maximum at $x = c$. It reaches an absolute minimum at $x = b$. Again, $f(x)$ has relative maxima at $x = a$ and $x = c$. Similarly, $f(x)$ has relative minima at $x = b$ and $x = d$.

It should be noted that a point on the graph of a function could be both a relative maximum (minimum) and an absolute maximum (minimum).

Thus, local maxima and minima can be determined from the first and second derivatives. The absolute maxima and minima can be found only by comparing the local maxima and local minima with the value of the function at the end points and by selecting the absolute maximum and minimum.

Local maxima and minima can be determined from the first and second derivatives.

Working Rules for Finding of Maximum, Minimum and Point of Inflection of a Function with Single Variable:

Step 1: Differentiate the given function and equate to zero and also find the roots.

(i.e. find $\frac{dy}{dx}$ and put $\frac{dy}{dx} = 0$. Calculate the stationary point.)

Step 2: Again differentiate the given function and put the values of roots in this second derivative function one by one (i.e. compute $\frac{d^2y}{dx^2}$ at these stationary points).

Step 3: If the second derivative is positive for a root then the given function is minimum. On the other hand, if the second derivative is negative for a root then the given function is maximum.

Illustrative Examples:

Example-1:

Let (i) $f(x) = 3x^2 - 14x + 5$

(ii) $f(x) = x^3 - 7x^2 + 6x - 2$

(iii) $f(x) = x^4 - 6x^3 + 4x^2 - 13$

Identify whether the above functions are increasing, decreasing or stationary at $x = 4$.

Solution:

(i) Given that, $f(x) = 3x^2 - 14x + 5$

$$\frac{d(f(x))}{dx} = f'(x) = 6x - 14$$

$$f'(4) = 6 \times 4 - 14 = 10 > 0$$

Thus, the function is increasing.

(ii) Given that, $f(x) = x^3 - 7x^2 + 6x - 2$

$$\frac{d(f(x))}{dx} = f'(x) = 3x^2 - 14x + 6$$

$$f'(4) = 3(4)^2 - 14(4) + 6 = -2 < 0$$

Thus, the function is decreasing.

(iii) Given that, $f(x) = x^4 - 6x^3 + 4x^2 - 13$

$$\frac{d(f(x))}{dx} = f'(x) = 4x^3 - 18x^2 + 8x$$

$$f'(4) = 4(4)^3 - 18(4)^2 + 8(4) = 0$$

Thus, the function is stationary.

Example-2:

Let (i) $f(x) = -2x^3 + 4x^2 + 9x - 15$

(ii) $f(x) = (5x^2 - 8)^2$

Identify whether the above functions are concave or convex at $x = 3$.

Solution:

(i) Given that, $f(x) = -2x^3 + 4x^2 + 9x - 15$

$$\frac{d(f(x))}{dx} = f'(x) = -6x^2 + 8x + 9$$

$$\frac{d^2(f(x))}{dx^2} = f''(x) = -12x + 8$$

$$f''(3) = -12(3) + 8 = -28 < 0$$

Thus, the function is concave.

(ii) Given that, $f(x) = (5x^2 - 8)^2$

$$\frac{d(f(x))}{dx} = f'(x) = 2(5x^2 - 8)(10x) = 100x^3 - 160x$$

$$f''(x) = 300x^2 - 160$$

$$f''(3) = 300(3)^2 - 160 = 2540 > 0$$

Thus, the function is convex.

Example-3:

Find the maximum and minimum values of the function: $x^4 + 2x^3 - 3x^2 - 4x + 4$

Solution:

Let $y = x^4 + 2x^3 - 3x^2 - 4x + 4$.

$$\frac{dy}{dx} = 4x^3 + 6x^2 - 6x - 4$$

Now, if $\frac{dy}{dx} = 0$

then, $4x^3 + 6x^2 - 6x - 4 = 0$

or, $2(x + 2)(2x + 1)(x - 2) = 0$

or, $x = -2, -1/2, 1$.

To find the maximum and minimum values we have to test these values in the second derivative of the function, which is $\frac{d^2y}{dx^2} = 12x^2 + 12x - 6$.

When $x = -2$, $\frac{d^2y}{dx^2} = 12(-2)^2 + 12(-2) - 6 = 66$, which is positive

Hence the given function attains minimum at $x = -2$

And the minimum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(2) = (-2)^4 + 2(-2)^3 - 3(-2)^2 - 4(-2) + 4 = 0$

When $x = -1/2$, $\frac{d^2y}{dx^2} = 12(-\frac{1}{2})^2 + 12(-\frac{1}{2}) - 6 = -9$, which is negative

Hence the given function attains maximum at $x = -1/2$

And the maximum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(-1/2) = (-\frac{1}{2})^4 + 2(-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 - 4(-\frac{1}{2}) + 4 = \frac{81}{61}$

When $x = 1$, $\frac{d^2y}{dx^2} = 12(1)^2 + 12(1) - 6 = 18$, which is positive

Hence the given function attains minimum at x =1.

And the minimum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(1) = (1)^4 + 2(1)^3 - 3(1)^2 - 4(1) + 4 = 0$

Example-4:

Show that the curve $y = x^2(3 - x)$ has a point of inflection at the point (1, 2).

Solution:

We are given that, $y = x^2(3 - x) = 3x^2 - x^3$.

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\frac{d^2y}{dx^2} = 6 - 6x$$

$$\frac{d^3y}{dx^3} = -6$$

For point of inflection, we must have, $\frac{d^2y}{dx^2} = 0$ *and* $\frac{dy^3}{dx^3} \neq 0$

Now, if $\frac{d^2y}{dx^2} = 0$

then, $6 - 6x = 0$

or, $x = 1$

And when $x = 1$, $\frac{dy^3}{dx^3} \neq 0$

When $x = 1$, then $y = 2$.

Hence (1, 2) is the point of inflection.

Example-5:

A sitar manufacturer notices that he can sells x sitars per week at p Taka each where $5x = 375 - 3p$. The cost of production is

$\left(500+13x+\frac{1}{5}x^2\right)$ Taka. Show that the maximum profit is obtained when the production is 30 sitars per week.

Solution:

Given that, $5x = 375 - 3p$

Thus, price $p = \frac{375-5x}{3}$

Revenue = price $\times$ quantity

$$= \frac{375-5x}{3} \times x$$

$$= \frac{375x-5x^2}{3}$$

Cost = $\left(500+13x+\frac{1}{5}x^2\right)$

Profit = Revenue – Cost

$$P = \frac{375x-5x^2}{3} - \left(500+13x+\frac{1}{5}x^2\right)$$

$$P = \frac{375x}{3} - \frac{5x^2}{3} - 500 - 13x - \frac{1x^2}{5}$$

Differentiate it with respect to *x*,

$$\frac{dP}{dx} = \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2x}{5}$$

For maxima and minima, $\frac{dP}{dx} = \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2x}{5} = 0$

$$x = 30.$$

Again,

$$\frac{d^2P}{dx^2} = -\frac{10}{3} - \frac{2}{5} = -\frac{56}{15} = -ve$$

Thus, the profit function is the maximum at $x = 30$.

Example-6:

A manufacturer sells x units of a product at a dollar price of $p = p(x) = 6565 - 10x - 0.1x^2$ per unit. The cost of manufacturing the product is

$C(x) = 0.05x^3 - 5x^2 + 20x + 250000, \quad 0 \le x \le 150$. How many units should be produced and sold to maximize the resulting profit?

Solution:

Total revenue = (Price per unit) × (Number of units sold)

$$R(x) = (6565 - 10x - 0.1x^2) \times x = 6565x - 10x^2 - 0.1x^3$$

Profit, $P(x)$ = Revenue – Cost

$$= (6565x - 10x^2 - 0.1x^3) - (0.05x^3 - 5x^2 + 20x + 250000)$$

$$= -0.15x^3 - 5x^2 + 6545x - 250000$$

To determine the quantity x that maximizes the profit function, we first find,

$$\frac{dP}{dx} = -0.45x^2 - 10x + 6545$$

Setting this first derivative equals to zero, we solve for critical values of x and $x = 110$ (Other root is outside of the domain).

Now $\frac{d^2P}{dx^2} = -0.9x - 10$

$= -(0.9)(110) - 10 = -109 < 0$; indicating that this critical value does indeed represent a maximum.

$P(110) = -0.15(110)^3 - 5(110)^2 + 6545(110) - 250000 = \$209800$.

Example-7:

A Company has examined its cost structure and revenue structure and has determined that C, the total cost, R, the total revenue and x, the number of units produced are related as: $C = 100 + 0.015x^2 \quad and \quad R = 3x$.

Find the production level x that will maximize the profits of the company. Find that profit. Find also the profit when $x = 120$.

Solution:

Profit, $P(x)$ = Revenue – Cost = $R - C$

$$P(x) = 3x - (100 + 0.015x^2) = 3x - 100 - 0.015x^2$$

Hence, $\frac{dP}{dx} = 3 - 0.030x$

For maximum or minimum, $\frac{dP}{dx} = 0$

$$3 - 0.030x = 0$$

$$x = 100 \text{ units}$$

Also $\frac{d^2P}{dx^2} = -0.030 < 0$

Thus, profit will be maximum when $x = 100$ and the maximum profit

$$P = 3\times 100 - 100 - 0.015\times(100)^2 = 50$$

When $x = 120$, Profit, $P = 3\times 120 - 100 - 0.015\times(120)^2 = 44$.

Example-8:

The total cost function of a firm is given by $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where C is the total cost and x is the output of the product. A tax at the rate of \$2 per unit of product is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where p is the price per unit of output, find the profit maximizing output and price.

Solution:

Profit, $P(x)$ = Total Revenue – Total Cost = TR – TC

$$P(x) = \{(2530 - 5x)x\} - \{(\frac{1}{3}x^3 - 5x^2 + 28x + 10) + 2x\}$$

$$= (2530x - 5x^2) - (\frac{1}{3}x^3 - 5x^2 + 30x + 10)$$

Hence, $\frac{dP}{dx} = (2530 - 10x) - (x^2 - 10x + 30)$.

For maximum or minimum, $\frac{dP}{dx} = 0$

$$(2530 - 10x) - (x^2 - 10x + 30) = 0$$

$$x^2 = 2500 \text{ units}$$

$x = 50$ (since production level cannot be negative, so ignoring –ve sign)

Also $\frac{d^2P}{dx^2} = -2x < 0$

Thus, profit will be maximum when $x = 50$ units and the maximum profit

$$P = 2530 - 5\times 50 = \$2280$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Identify whether the following function is increasing, decreasing or stationary at $x = 5$. Find also identify whether the function is concave or convex at $x = 5$.
   $$f(x) = 2x^3 - 30x^2 + 126x + 59.$$

2. A firm has determined that its weekly profit function is given by
   $$P(x) = 95x - 0.05x^2 - 5000; \qquad 0 \le x \le 1000$$
   where $P(x)$ is the profit in dollars and x is the number of units of the product sold.
   For what value of x does profit reach a maximum? What is this maximum profit?

3. A study has shown that the cost of producing Orange Juice of a manufacturing concern is given by $C = 30 + 1.5x + 0.0008x^2$. What is the marginal cost at $x = 1000$?

   If the Juice sells for Tk.5 each for what values of x does marginal cost equal marginal revenue? [Hint: Marginal cost is the value of dC/dx at $x = 1000$.]

Lesson-2: Optimization of Multivariate Functions

After completing this lesson, you should be able to:

- Express the concept of optimization with multivariate function;
- Determine the maximum and minimum point of a multivariate function.

Introduction:

Many economic activities involve functions of more than one independent variable. Let $Z = f(x, y)$ and $P = f(x, y, z)$ are defined as functions of two and three independent variables respectively. In order to measure the effect of a change in a single independent variable on the dependent variable in a multivariate function, the partial derivative is needed. Partial derivative with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant. This lesson extends the ideas of relative maximum and minimum for functions of one variable to multivariate functions.

Partial derivative with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant.

Determination of the Maximum and Minimum Values of a Function with Two Independent Variables:

Let $\phi(x, y)$ be a function of two independent variables *x* and *y*. We are to investigate at (a,b) whether $\phi(x, y)$ is maximum or minimum.

For the existence of a maximum or a minimum at (a,b)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$$

Again let, $A = \dfrac{\partial^2\phi}{\partial x^2}$, $H = \dfrac{\partial^2\phi}{\partial x\partial y}$, $B = \dfrac{\partial^2\phi}{\partial y^2}$.

Case 1: If $AB - H^2$ is positive. $\phi(a,b)$ is maximum or minimum if *A* and *B* are both negative or are both positive respectively.

Case 2: If $AB - H^2$ is negative, $\phi(a,b)$ is neither a maximum nor a minimum.

Case 3: If $AB < H^2$ when *A* and *B* have the same signs, the function is at an inflection point; when *A* and *B* have different signs, the function is at a saddle point.

Case 4: If $AB = H^2$, the test is inconclusive.

Illustrative Examples:

Example-1:

Determine the values of *x* and *y* for which, $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$ is an optimum; specify whether the optimum is a maximum or minimum. Calculate the value of the function at the optimum.

Solution:

Given that, $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$

$$\frac{\partial z}{\partial x} = 8x - 4y + 10 = 0 \quad \text{.....................(i)}$$

$$\frac{\partial z}{\partial y} = -4x + 4y - 6 = 0 \quad \text{...................(ii)}$$

Solving equations (i) and (ii) simultaneously for *x* and *y* gives $x = -1$ and $y = \frac{1}{2}$

Again, $A = \frac{\partial^2 z}{\partial x^2} = 8$, $B = \frac{\partial^2 z}{\partial y^2} = 4$, $H = \frac{\partial^2 z}{\partial x \partial y} = -4$

$AB - H^2 = (8)(4) - (-4)^2 = 16 > 0$.

Since the second derivatives are positive and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative minimum at $x = -1$ and $y = \frac{1}{2}$.

The value of the function $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$ at the optimum is,

$$Z(-1, \frac{1}{2}) = 4(-1)^2 + 2(\frac{1}{2})^2 + 10(-1) - 6(\frac{1}{2}) - 4(-1)(\frac{1}{2}) = -6\frac{1}{2}$$

Example-2:

Determine the values of *x* and *y* for which, $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$ is an optimum; specify whether the optimum is a maximum or minimum. Calculate the value of the function at the optimum.

Solution:

Given that, $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$

$$\frac{\partial z}{\partial x} = -8x + 4y + 16 = 0 \quad \text{.......................(i)}$$

$$\frac{\partial z}{\partial y} = 4x - 4y - 12 = 0 \quad \text{........................(ii)}$$

Solving equations (i) and (ii) simultaneously for *x* and *y* gives $x = 1$ and $y = -2$

Again, $A = \frac{\partial^2 z}{\partial x^2} = -8$, $B = \frac{\partial^2 z}{\partial y^2} = -4$, $H = \frac{\partial^2 z}{\partial x \partial y} = 4$

$AB - H^2 = (-8)(-4) - (4)^2 = 16 > 0$.

Since the second derivatives are negative and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative maximum at $x = 1$ and $y = -2$.

The value of the function $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$ at the optimum is:

$$Z(1, -2) = -4(1)^2 + 4(1)(-2) - 2(-2)^2 + 16(1) - 12(-2) = 20$$

Example-3:

The total production cost of a product is given by $f(x, y) = x^2 + y^2 - 5x - 9y - xy + 90$.

where $f(x, y)$ = Total cost in thousand dollar.

x = Number of labor hours used (in hundred).

y = Number of pounds for raw materials used (in hundred).

It is required to determine how many labor hours and how many pounds of raw materials should be used in order to minimize the total cost.

Solution:

Given that, $f(x, y) = x^2 + y^2 - 5x - 9y - xy + 90$

The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x - 5 - y = 0 \quad \text{..................(i)}$$

$$\frac{\partial f}{\partial y} = 2y - 9 - x = 0 \quad \text{.................(ii)}$$

Solving equations (i) and (ii) simultaneously for x and y gives $x = \frac{19}{3}$ and $y = \frac{23}{3}$

Again, the second order partial derivatives are, $A = \frac{\partial^2 f}{\partial x^2} = 2$, $B = \frac{\partial^2 f}{\partial y^2} = 2$, $H = \frac{\partial^2 f}{\partial x \partial y} = -1$

$AB - H^2 = (2)(2) - (-1)^2 = 3 > 0$.

Since the second derivatives are positive and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative minimum at $x = \frac{19}{3}$ and $y = \frac{23}{3}$.

The total cost will be minimum when $^{19}/_3$ hundred (633) labor hours and $^{23}/_3$ hundred (767) pounds of raw materials are used. The total cost with this production strategy is:

$$f\left(\frac{19}{3}, \frac{23}{3}\right) = \left(\frac{19}{3}\right)^2 + \left(\frac{23}{3}\right)^2 - 5\left(\frac{19}{3}\right) - 9\left(\frac{23}{3}\right) - \left(\frac{19}{3}\right)\left(\frac{23}{3}\right) + 90$$

$= 39.667$ thousand or 39,667.

Determination of the Maximum and Minimum Values of a Function with Three Independent Variables:

Let $\phi(x, y, z)$ be a function of three independent variables *x*, *y* and *z*. We are to investigate at (a, b, c) whether $\phi(x, y, z)$ is maximum or minimum.

For the existence of a maximum or a minimum at (a, b, c)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$$

Again let $A = \frac{\partial^2 \phi}{\partial x^2}$, $B = \frac{\partial^2 \phi}{\partial y^2}$, $C = \frac{\partial^2 \phi}{\partial z^2}$, $F = \frac{\partial^2 \phi}{\partial y \partial z}$, $G = \frac{\partial^2 \phi}{\partial z \partial x}$,

$$H = \frac{\partial^2 \phi}{\partial x \partial y}$$

Case 1: $\phi(a, b, c)$ is minimum if *A*, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are all positive.

Case 2: $\phi(a, b, c)$ is maximum if *A*, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are alternately negative and positive.

Case 3: If the above conditions are not satisfied, then $\phi(a, b, c)$ is neither a maximum nor a minimum.

Illustrative Examples:

Example-4:

Show that the function $\phi(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$ has a minimum value at $(-2/3, -1/3, 1)$

Solution:

We are given that, $\phi(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$.

$$\frac{\partial \phi}{\partial x} = 2x + 1 - y.$$

$$\frac{\partial \phi}{\partial y} = 2y - x$$

$$\frac{\partial \phi}{\partial z} = 2z - 2.$$

$\phi\ (x, y, z)$ will be maximum or minimum if $\frac{\partial \phi}{\partial x} = 0$, $\frac{\partial \phi}{\partial y} = 0$, $\frac{\partial \phi}{\partial z} = 0$.

That is $\dfrac{\partial \phi}{\partial x} = 2x + 1 - y = 0$ ....(i)

$$\frac{\partial \phi}{\partial y} = 2y - x = 0 \quad \text{................(ii)}$$

$$\frac{\partial \phi}{\partial z} = 2z - 2 = 0 \quad \text{..................(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously, we get $x = -{}^{2}/_{3}$, $y = -{}^{1}/_{3}$ and $z = 1$.

Again, $A = \dfrac{\partial^2 \phi}{\partial x^2} = 2$, $B = \dfrac{\partial^2 \phi}{\partial y^2} = 2$, $C = \dfrac{\partial^2 \phi}{\partial z^2} = 2$, $F = \dfrac{\partial^2 \phi}{\partial y \partial z} = 0$,

$$G = \frac{\partial^2 \phi}{\partial z \partial x} = 0,\ H = \frac{\partial^2 \phi}{\partial x \partial y} = -1.$$

Since, $A = 2$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ and

$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$ are all positive.

Hence, the given function $\phi(x, y, z)$ is minimum at $(-2/3, -1/3, 1)$.

Example-5:

Optimize the following function:

$$y = -5x_1^2 + 10x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$$

Solution:

We are given that,

$\phi(x_1, x_2, x_3) = y = -5x_1^2 + 10x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$

$$\frac{\partial \phi}{\partial x_1} = -10x_1 + 10 + x_3 = 0.$$

$$\frac{\partial \phi}{\partial x_2} = -4x_2 + 2x_3 + 4 = 0$$

$$\frac{\partial \phi}{\partial x_3} = x_1 + 2x_2 - 8x_3 = 0.$$

Solving the above three equations by using Cramer's Rule we get $x_1 = 1.04$, $x_2 = 1.22$ and $x_3 = 0.43$.

Again, $A = \frac{\partial^2 \phi}{\partial x_1^2} = -10$, $B = \frac{\partial^2 \phi}{\partial x_2^2} = -4$, $C = \frac{\partial^2 \phi}{\partial x_3^2} = -8$,

$F = \frac{\partial^2 \phi}{\partial x_2 \partial x_3} = 2$,

$$G = \frac{\partial^2 \phi}{\partial x_3 \partial x_1} = 1, \; H = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0.$$

Since, $A = -10$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -10 & 0 \\ 0 & -4 \end{vmatrix} = 40$ and

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -10 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -8 \end{vmatrix} = -276$$

Since the principal minors alternate correctly in sign, hence the given function is maximized at $x_1 = 1.04$, $x_2 = 1.22$ and $x_3 = 0.43$.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the maxima and minima for the following functions:

   (a) $f(x, y) = 2x^2 - 4x + y^2 - 4y + 4$.

   (b) $f(x, y) = 2x^2 - 4x + 8y^2 + 80y + 50xy + 100$.

2. A company produces two products, x units of type A and y units of type B per month. If the revenue and cost equations for the month are given by $R(x, y) = 11x + 14y$ and $C(x, y) = x^2 - xy + 2y^2 + 3x + 4y + 10$, find the state (x, y) that yields maximum profit.

3. The cost of construction C of a project depends upon number of skilled workers *(x)* and unskilled workers *(y)*. It is given that cost $C(x, y) = 40{,}000 + 9x^3 - 72xy + 9y^2$. Determine the number of skilled and unskilled workers results in minimum cost. Find also the minimum cost.

4. The yearly profits of a small service organization Econo Ltd. are dependent upon the number of workers *(x)* and the number of units of advertising *(y)*, and the profit function is, $P(x, y) = 412x + 806y - x^2 - 5y^2 - xy$. Determine the number of workers and the number of units in advertising that results in maximum profits. Determine also the maximum profits.

Lesson-3: Constrained Optimization with Lagrangian Multipliers

After completing this lesson, you should be able to:

- Express the concept of constrained optimization;
- Determine the maximum point of a function with some constraints;
- Determine the minimum point of a function with some constraints.

Introduction

A decision maker must consider different types of physical and legal restrictions.

Decision makers do not normally have unlimited resources for their use. A decision maker must consider different types of physical and legal restrictions. Solutions to economic problems often have to be found under constraints, e.g., maximizing utility subject to a budget constraint or minimizing costs subject to some such minimal requirement of output as a production quota etc. Classical differential calculus is used to optimize (maximize or minimize) a function subject to constraint. In this respect, use of the Lagrangian function greatly facilitates this task.

Lagrangian multipliers provide a method of determining the optimum value of a differentiable nonlinear function subject to linear or nonlinear constraints.

The Lagrangian multiplier (λ) approximates the marginal impact on the objective function caused by a small change in the constant of the constraint. Lagrangian multipliers provide a method of determining the optimum value of a differentiable nonlinear function subject to linear or nonlinear constraints. The method of Lagrangian multipliers is useful in allocating scarce resources between alternative uses. In this lesson, we shall now consider procedures used for determining relative maxima and minima for a multivariate function on which certain constraints are imposed.

Working Rule for Constrained Optimization with Lagrange Multiplier:

Step 1: Given a function $f(x, y)$ subject to a constraint $g(x, y) = K$ (a constant), a new function F can be formed by setting the constraint equal to zero.

Step 2: Multiplying it by λ (the Lagrange multiplier) and adding the product to the original function:

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

Step 3: Critical values x, y, λ at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero and solving simultaneously:

$$F_x(x, y, \lambda) = 0,\ F_y(x, y, \lambda) = 0,\ F_\lambda(x, y, \lambda) = 0,$$

Illustrative Examples:

Example -1:

Determine the critical points and the constrained optimum for $Z = 4x^2 - 2xy + 6y^2$; subject to $x + y = 72$.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

$$= 4x^2 - 2xy + 6y^2 + \lambda(72 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 8x - 2y - \lambda = 0\,.$$

$$\frac{\partial F}{\partial y} = -2x + 12y - \lambda = 0\,.$$

$$\frac{\partial F}{\partial \lambda} = 72 - x - y = 0\,.$$

The above three equations are solved simultaneously and we get $x = 42, y = 30$ and $\lambda = 276$.

Thus Z = [4 $(42)^2$ – 2 (42) (30) + 6 $(30)^2$ + 276 (72 – 42 – 30)] = 9936.

With the Lagrangian multiplier λ = 276 means that one unit increase in the constant of the constraint will lead to an increase of approximately 276 in the value of the objective function and Z $\approx$ 10212.

Example-2:

Determine the critical points and the constrained optima for $Z = x^2 + 3xy + y^2$; subject to $x + y = 100$.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

$$= x^2 + 3xy + y^2 + \lambda(100 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 2x + 3y - \lambda = 0\,.$$

$$\frac{\partial F}{\partial y} = 3x + 2y - \lambda = 0\,.$$

$$\frac{\partial F}{\partial \lambda} = 100 - x - y = 0\,.$$

The above three equations are solved simultaneously and we get $x = 50, y = 50$ and $\lambda = -250$. Thus, the constrained optimum,

Z = [$(50)^2$ + 3 (50) (50) + $(50)^2$ - 250 (100 – 50 – 50)] = 12500.

To determine if the function reaches a maximum or a minimum, we evaluate the function at points adjacent to x = 50 and y = 50. The function is a constrained maximum since adding Δx and Δy to the function in both directions gives a functional value less than the constrained optimum.

That is F (49, 51, –250) = 12,499

F (51, 49, –250) = 12,499.

Example-3:

What output mix should a profit maximizing firm produce when its total profit function is $\pi = 80x - 2x^2 - xy - 3y^2 + 100y$ and its maximum output capacity is $x + y = 12$? Estimate also the effect on profits if output capacity is expanded by one unit.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

$$= 80x - 2x^2 - xy - 3y^2 + 100y + \lambda(12 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 80 - 4x - y - \lambda = 0\,.$$

$$\frac{\partial F}{\partial y} = -x - 6y + 100 - \lambda = 0\,.$$

$$\frac{\partial F}{\partial \lambda} = 12 - x - y = 0\,.$$

The above three equations are solved simultaneously and we get $x = 5, y = 7$ and $\lambda = 53$.

Thus profit, $\pi = [80(5) - 2(5)^2 - (5)(7) - 3(7)^2 + 100(7) = 868$.

Hence, λ=53 means that an increase in output capacity should lead to increased profits of approximately 53.

Inequality Constraints:

The method of Lagrangian multipliers can be modified to incorporate constraints that take the form of inequalities rather than equalities.

The method of Lagrangian multipliers can be modified to incorporate constraints that take the form of inequalities rather than equalities. The problem now becomes that of determining the extreme points of the multivariate function $z = f(x, y)$ subject to the inequality $g(x, y) \le 0$ or $g(x, y) \ge 0$. We shall now a relatively simple extension of the Lagrangian technique that provides a solution to the problem of optimizing an objective function subject to single constraining inequality. The general problem of optimizing an objective function subject to n inequalities is not considered in this lesson.

In the problem of optimizing an objective function subject to a constraining inequality, two cases must be considered. Firstly, the constraining equality may act as an upper or lower bound on the function. Secondly, the constraining inequality does not act as an upper or lower bound on the function. For either of the above two cases, the method of optimizing a function subject to a constraining inequality is to assume that the constraining inequality is equality; i.e. assume $g(x, y) = 0$. The inequality is changed to equality and the critical values and constrained optimum are obtained by using the method of Lagrangian multipliers. The sign of the Lagrangian multiplier is used to determine whether the constraint is actually limiting the optimum value of the objective function. The procedure is as follows:

Case 1: When the constraining inequality $g(x, y) \leq 0$:

For maximizing the objective function

(i) If $\lambda > 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

(ii) If $\lambda \leq 0$, the restriction acts as an upper bound and the result can be obtained by assuming that $g(x, y) = 0$ is the constrained optimum.

For minimizing the objective function

(i) If $\lambda > 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

(ii) If $\lambda \leq 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

Case 2: When the constraining inequality $g(x, y) \geq 0$:

For maximizing the objective function

(i) If $\lambda > 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

(ii) If $\lambda \leq 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

For minimizing the objective function

(i) If $\lambda > 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

(ii) If $\lambda \leq 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

The following examples illustrate this technique.

Illustrative Example:

Example-4:

Determine the maximum of the function $z = x^2 + 3xy + y^2$ subject to $x + y \leq 100$.

Solution:

Given that the objective function is $z = x^2 + 3xy + y^2$ ....(i)

the constraint equation is $x + y \leq 100$. ....(ii)

To determine the solution we treat the inequality as equality and form the Lagrangian expression

$$F(x, y, \lambda) = x^2 + 3xy + y^2 + \lambda[x + y - 100]$$

The critical values are $x^* = 50$, $y^* = 50$ and $\lambda^* = -250$.

Since the Lagrangian multiplier is negative, an increase in the constant of the constraining function would increase the value of the objective function. Therefore, the value of the function is limited by the constraint.

Example-5:

Determine the maximum profit of the following profit function $z = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x$ subject to $x + y \geq 5$. Determine the values of *x* and *y* that maximizes the profits. Assume that *x* and *y* in million dollar and profits in thousand dollar respectively.

Solution:

Given that the objective function is

$$z = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x \text{ ....(i)}$$

and the constraint equation is $x + y \geq 5$. ....(ii)

The Lagrangian expression is

$$F(x, y, \lambda) = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x + \lambda[x + y - 5]$$

The partial derivatives of this expression are equal to zero.

$$\frac{\partial F}{\partial x} = -8x + 2y + 12 + \lambda = 0 \text{ ....(iii)}$$

$$\frac{\partial F}{\partial y} = 2x - 12y + 20 + \lambda = 0 \text{ ....(iv)}$$

$$\frac{\partial F}{\partial \lambda} = x + y - 5 = 0 \text{ ....(v)}$$

Solving equations (iii), (iv) and (v) simultaneously we get,

$$x^* = 2.584, \quad y^* = 2.416 \quad \text{and} \quad \lambda^* = 3.834.$$

.

Thus, the maximum profit will be z = [600 – 4(2.584)^2 + 20(2.416) + 2 (2.584)(2.416) – 6(2.416)^2 + 12(2.584)] = 630.083 thousand or $630,083. Profits, subject to the constraint, are maximized when *x* = $2.584 million ($2584000), *y* = $2.416 million ($2416000) are obtained. Since λ is positive, a decrease in the constraint constant will increase the value of the objective function. Therefore, the constraint acts as a lower bound on the values of the variables.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Determine the critical points and the constrained optimum for $f(x, y) = 12xy - 3y^2 - x^2$ subject to $x + y = 16$.
2. Determine the critical points and the constrained optimum for $f(x, y) = 3x^2 + 4y^2 - xy$ subject to $2x + y = 21$.
3. Determine the maximum of the function $z = 10xy - 5x^2 - 7y^2 + 40x$ subject to $x + y \leq 12$.
4. Determine the minimum of the function $z = 4x^2 + 5y^2 - 6y$ subject to $x + 2y \geq 20$.
5. A firm's total costs can be presented as $C = 3x^2 + 5xy + 6y^2$, find the firm's minimum cost to meet a production quota of $5x + 7y = 732$.